This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Math 1808.81.3

SIGILLVM·ACADEMIAE·HARVARDIANAE·IN·NOV·ANG·

SCIENCE CENTER LIBRARY

FROM

ELEMENTS

OF

QUATERNIONS.

BY

A. S. HARDY, Ph. D.,

PROFESSOR OF MATHEMATICS, DARTMOUTH COLLEGE.

BOSTON:
PUBLISHED BY GINN, HEATH, & CO.
1881.

Math 18[illegible].71.3

✓

Harvard College Library
Aug. 28, 191
From Estate of
Prof. Charles J. White
Cambridge

Entered according to Act of Congress, in the year 1881, by
A. S. HARDY,
in the office of the Librarian of Congress, at Washington.

GINN, HEATH, & CO.
J. S. CUSHING, PRINTER, 16 HAWLEY STREET,
BOSTON.

PREFACE.

THE object of the following treatise is to exhibit the elementary principles and notation of the Quaternion Calculus, so as to meet the wants of beginners in the class-room. The *Elements* and *Lectures* of Sir William Rowan Hamilton, while they may be said to contain the suggestion of all that will be done in the way of Quaternion research and application, are not, for this reason, as also on account of their diffuseness of style, suitable for the purposes of elementary instruction. Tait's work on Quaternions is also, in its originality and conciseness, beyond the time and needs of the beginner. In addition to the above, the following works have been consulted:

Calcolo dei Quaternione. Bellavitis; Modena, 1858.

Exposition de la Méthode des Équipollences. Traduit de l'Italien de Giusto Bellavitis, par C.-A. Laisant; Paris, 1874. (Original memoir in the *Memoirs of the Italian Society.* 1854.)

Théorie Élémentaire des Quantités Complexes. J. Hoüel; Paris, 1874.

Essai sur une Manière de Representer les Quantités Imaginaires dans les Construction Géométriques. Par R. Argand; Paris, 1806. Second edition, with preface by J. Hoüel; Paris, 1874. Translated, with notes, from the French, by A. S. Hardy. Van Nostrand's Science Series, No. 52; 1881.

Kurze Anleitung zum Rechnen mit den (Hamilton'schen) Quaternionen. J. Odstrčil; Halle, 1879.

Applications Mécaniques du Calcul des Quaternions. Laisant; Paris, 1877.

Introduction to Quaternions. Kelland and Tait; London, 1873.

A free use has been made of the examples and exercises of the last work; and, in Article 87, is given, by permission, the substance of a paper from Volume I., page 379, *American Journal of Mathematics*, illustrating admirably the simplicity and brevity of the Quaternion method.

If this presentation of the principles shall afford the undergraduate student a glimpse of this elegant and powerful instrument of analytical research, or lead him to follow their more extended application in the works above cited, the aim of this treatise will have been accomplished.

The author expresses his obligation to Mr. T. W. D. Worthen for valuable assistance in the preparation of this work, and to Mr. J. S. Cushing for whatever of typographical excellence it possesses.

A. S. HARDY.

HANOVER, N.H., June 21, 1881.

CONTENTS.

CHAPTER I.

Addition and Subtraction of Vectors; or, Geometric Addition and Subtraction.

Article. Page.

1. Definition of a *vector*. Effect of the minus sign before a vector . . . 1
2. Equal vectors . . . 2
3. Unequal vectors. Vector addition . . . 2
4. Vector addition, commutative . . . 3
5. Vector addition, associative . . . 3
6. Transposition of terms in a vector equation . . . 4
7. Definition of a *tensor* . . . 4
8. Definition of a *scalar* . . . 5
9. Distributive law in the multiplication of vector by scalar quantities . . . 6
10. If $\Sigma\alpha + \Sigma\beta = 0$, then $\Sigma\alpha = 0$ and $\Sigma\beta = 0$ . . . 7
11. Examples . . . 8
12. Complanar vectors. Condition of complanarity . . . 15
13. Co-initial vectors. Condition of collinearity . . . 16
14. Examples . . . 17
15. Expression for a medial vector . . . 24
16. Expression for an angle-bisector . . . 25
17. Examples . . . 26
18. Mean point . . . 28
19. Examples . . . 28
20. Exercises . . . 30

CHAPTER II.

Multiplication and Division of Vectors; or, Geometric Multiplication and Division.

21. Elements of a *quaternion* . . . 32
22. Equal quaternions . . . 34
23. Positive rotation . . . 35

Article. Page.
24. Analytical expression for a quaternion. Product and quotient of rectangular unit-vectors. Tensor and *versor* of a quaternion 36
25. Symbolic notation $q = \mathbf{T}q\mathbf{U}q$ 39
26. Reciprocal of a quaternion 39
27. Quadrantal versors, i, j, k 40
28. Whole powers of unit vectors. Square of a unit vector is -1, 41
29. Associative law in the multiplication of rectangular unit-vectors 42
30. Negative sign commutative with i, j, k 43
31. Commutative law not true for the products of i, j, k 43
32. Reciprocal of a unit vector 44
33. A unit vector commutative with its reciprocal. Reciprocal of any vector 45
34. Product and quotient of any two rectangular vectors 47
35. Square of any vector 47
36. Distributive law true of the products of i. j. k 48
37. Exercises 48
38. Symbolic notation, $q = \mathbf{S}q + \mathbf{V}q$ 49
39. De Moivre's theorem 50
40. Products of two vectors. Symbolic notation 56
41. General principles and formulae 58
42. Powers of vectors and quaternions 61
43. Relation between the vector and Cartesian determination of a point 64
44. Right, complanar, diplanar and collinear quaternions. Any two quaternions reducible to the forms $\frac{\alpha}{\beta}, \frac{\gamma}{\beta}$ and $\frac{\alpha}{\gamma}, \frac{\gamma}{\delta}$ . . 65
45. Reciprocal of a vector, scalar and quaternion 66
46. Conjugate of a vector, scalar and quaternion 68
47. Opposite quaternions 70
48. $\mathbf{UK}q = \mathbf{U}\frac{1}{q} = \frac{1}{\mathbf{U}q} = \mathbf{KU}q$ 71
49. Representation of versors by spherical arcs 71
50. Addition and subtraction of quaternions. **K, S** and **V** distributive symbols. **K** commutative with **S** and **V**. **T** and **U** not distributive symbols 72
51. Multiplication of quaternions; not commutative. $\mathbf{U}\Pi q = \Pi\mathbf{U}q$. $\mathbf{T}\Pi q = \Pi\mathbf{T}q$. $\mathbf{K}qr = \mathbf{K}r\mathbf{K}q$. Product or quotient of complanar quaternions 74
52. Distributive and associative laws in quaternion and vector multiplication 79
53. General formulae 82

Article. Page.
54. Applications 82
55. Formulae relating to the products of two or more vectors . . 99
56. Exercises 106
57. Examples. Applications to Spherical Trigonometry 108
58. General Formulae 119
59. Applications to Plane Trigonometry 125

CHAPTER III.

Applications to Loci.

60. General equations of a line and surface 131
61. Use of Cartesian forms in conjunction with quaternion reasoning 132
62. Non-commutative law in quaternion differentiation. Differentiation of scalar functions 132
63. Quaternion differentiation 134
64. Illustration 136
65. Distributive principle 138
66. Quadrinomial form 139
67. Examples 140

The Right Line.

68. Right line through the origin 145
69. Parallel lines 145
70. Right line through two given points 146
71. Perpendicular to a given line; its length 147
72. Equations of a right line are linear and involve one independent variable scalar 150

The Plane.

73. Equation of a plane 151
74. Plane making equal angles with three given lines 152
75. Plane through three given points 153
76. Equations of a plane are linear and involve two independent variable scalars 154
77. Exercises and problems on the right line and plane 154

The Circle and Sphere.

78. Equations of the circle 164
79. Equations of the sphere 165

Article. Page.
80. Tangent line and plane 166
81. Chords of contact 167
82. Exercises and problems on the circle and the sphere 167
83. Exercises in the transformation and interpretation of elementary symbolic forms 176

The Conic Sections. Cartesian Forms.

84. The parabola 178
85. Tangent to the parabola 178
86. Examples on the parabola 180
87. Relations between three intersecting tangents to the parabola 185
88. The ellipse 191
89. Examples on the ellipse 192
90. The hyperbola 195
91. Examples on the hyperbola 195

---

92. Linear equation in quaternions. Conjugate and self-conjugate functions 199
93. General equations of the conic sections 201
94. The ellipse 204
95. Examples 206
96. The parabola 214
97. Examples 216
98. The cycloid 222
99. Elementary applications to mechanics 223
100. Miscellaneous Examples 231

ELEMENTS OF QUATERNIONS.

QUATERNIONS.

CHAPTER I.

Addition and Subtraction of Vectors, or Geometric Addition and Subtraction.

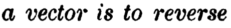
1. A Vector *is the representative of transference through a given distance in a given direction.*

Thus, if A, B are any two points, *vector* AB implies a translation from A to B.

A vector may be represented geometrically by a right line, whose length denotes the distance over which transference takes place, and whose direction denotes the direction of the transference. In thus designating a vector, the direction is indicated by the *order* of the letters.

Thus, AB (Fig. 1) denotes transference from A to B, and BA from B to A.

Fig. 1.



Retaining the algebraic signification of the signs + and −, if AB denotes motion from A to B, then −AB will denote motion from B to A, and

$$AB = -BA, \quad -AB = BA \quad . \quad . \quad . \quad . \quad . (1).$$

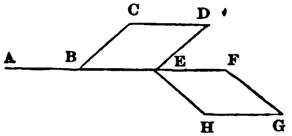
Hence; *the effect of a minus sign before a vector is to reverse its direction.*

The conception of a vector, therefore, implies that of its two elements, *distance* and *direction;* it was first defined as a *directed right line.* It is now applied more generally to all quantities determined by magnitude and direction. Thus, force, the path of a moving body, velocity, an electric current, etc., are vector quantities.

Analytically, vectors are represented by the letters of the Greek alphabet, α, β, γ, etc.

2. It follows, from the definition of a vector, that *all lines which are equal and parallel may be represented by the same vector symbol with like or unlike signs.* If equal and drawn in the same direction, they will have the same sign. Hence an equality between two vectors implies equality in distance with the same direction.

Fig. 2.



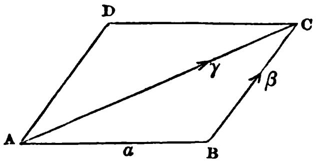
Thus, if AB (Fig. 2), CD, BE, EF and HG are equal and drawn in the same direction, they may be represented by the same vector symbol, and

$$AB = CD = BE = EF = HG = \alpha \quad . \quad . \quad . \quad . \quad (2).$$

3. It follows also from the definition of a vector that, if vectors are not parallel, they cannot be represented by the same vector symbol.

Thus, if the point A (Fig. 3) move over the right line AB, from A to B, and then over the right line BC, from B to C, and $AB = \alpha$, BC must be denoted by some other symbol, as β.

Fig. 3.



The result of these two successive translations of the point A is the same as that of the single and direct translation $AC = \gamma$, from A to C; in either case A is found at the extremity of the diagonal of the parallelogram of which AB and BC are the sides. This combination of successive translations is called *addition*, and is written in the ordinary way,

$$\alpha + \beta = \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This expression would be absurd if the symbols denoted magnitudes only. It means that transference from A to B, followed by transference from B to C, is equivalent to transference from A to C. The sign + does not therefore denote a numerical addition, or the sign = an equality between magnitudes. It is, however, called an equation, and read, as usual, "α plus β is equal to γ." This kind of addition is called *geometric addition.*

4. If the point A (Fig. 3), instead of moving over the sides AB, BC of the parallelogram ABCD, had moved in succession over the other two sides, AD and DC, the result would still have been the same as that of the single translation over the diagonal AC. But since AB and BC are equal in length to DC and AD respectively, and are drawn in the same direction, we have (Art. 2)

$$AB = DC \quad \text{and} \quad BC = AD,$$

and if the first two translations are represented by AB and BC, the second two may be represented by BC and AB, or

$$\alpha + \beta = \beta + \alpha = \gamma \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence *the operation of vector addition is commutative*, or the sum of any number of given vectors is independent of their order.

5. If the point A (Fig. 4) move in succession over the three edges AB, BC, CG of a parallelopiped, we have

$$AB + BC = AC,$$

and

$$AC + CG = AG,$$

or

$$(AB + BC) + CG = AG.$$

In like manner

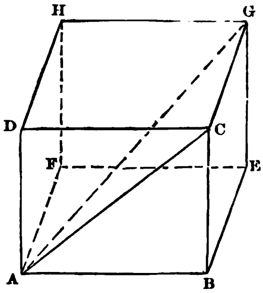
$$BC + CG = BG,$$

$$AB + BG = AG,$$

or

$$AB + (BC + CG) = AG.$$

Fig. 4.



Hence

$$(AB + BC) + CG = AB + (BC + CG) \quad . \quad . \quad . \quad (5),$$

and *the operation of vector addition is associative*, or the sum of any number of given vectors is independent of the mode of grouping them.

6. Since, if AC $= \gamma$ (Fig. 3), then CA $= -\gamma$, we have

$$\alpha + \beta - \gamma = 0,$$

or, comparing with equation (3),

$$\alpha + \beta = \gamma,$$

a term may be transposed from one member to another in a vector equation by changing its sign.

Also, in every triangle, any side may be considered as the sum or difference of the other two, depending upon their directions as vectors. Thus (Fig. 3)

$$\gamma - \beta = \alpha,$$
$$\gamma - \alpha = \beta.$$

It is to be observed that no one direction is assumed as positive, as in Cartesian Geometry. The only assumption is that opposite directions shall have opposite signs. The results must, of course, be interpreted in accordance with the primitive assumptions. Thus, had we assumed BA $= \alpha$ (Fig. 3), γ and β being as before, then

$$\beta - \alpha = \gamma,$$
$$\alpha - \beta = -\gamma.$$

7. If two vectors having the same direction be added together, the sum will be a vector in the same direction. If the vectors be also equal in length, the length of the vector sum will be the sum of their lengths. If n vectors, of equal length and drawn in the same direction, be added together, the sum will be the product of one of these vectors by n, or a vector having the same direction and whose length is n times the common length. If then (Fig. 2)

$$\text{AF} = x\text{AB} = x\text{CD} = x\alpha,$$

where A, B and F are in the same straight line, CD = AB, and x is a positive whole number, x expresses the ratio of the lengths of AF and α. From the case in which x is an integer we pass, by the usual reasoning, to that in which it is fractional or incommensurable. Vectors, then, in the same direction, have the same ratio as the corresponding lengths.

If $\mathrm{AB} = \alpha$ be assumed as the unit vector, then

$$\mathrm{AF} = m\alpha,$$

in which m is a *positive* numerical quantity and is called the **Tensor**. It is the ratio of the length of the vector $m\alpha$ to that of the unit vector α, or the numerical factor by which the unit vector is multiplied to produce the given vector.

Any vector, as β, may be written in general notation

$$\beta = \mathrm{T}\beta\,\mathrm{U}\beta.$$

In this notation, $\mathrm{T}\beta$ (read "tensor of β") is the numerical factor which stretches the unit vector so that it shall have the proper length; hence its name, tensor. It is, strictly speaking, an abstract number without sign, but, to distinguish between it and the negative of algebra, it may be said to be always positive. $\mathrm{U}\beta$ (read "versor of β") is the unit vector having the direction of β; the reason for the name versor will appear later.

p, 38

T and U are also general symbols of operation. Written before an expression, they denote the operations of taking the tensor and versor, respectively. Thus, if the length of β is n times that of the unit vector,

$$\mathrm{T}(\beta) = n,$$

where T denotes the operation of taking the stretching factor, *i.e.* the tensor. While

$$\mathrm{U}(\beta) = \mathrm{U}\beta$$

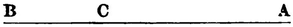
indicates the operation of taking the unit vector, that is, of reducing a vector β to its unit of length without changing its direction.

8. If BC (Fig. 5) be any vector, and $\mathrm{BA} = y\mathrm{BC}$, then

$$-\mathrm{BA} = \mathrm{AB} = -y\mathrm{BC};$$

and, in general, if BA and BC be any two real vectors, *parallel* and of unequal length, we may always conceive of a coefficient y which shall satisfy the equation

Fig. 5.



$$\mathrm{BA} = y\mathrm{BC},$$

where y is plus or minus, according as the vectors have the same or opposite directions. y may be called the geometric quotient, and is a real number, plus or minus, expressing numerically the ratio of the vector lengths. This quotient of *parallel* vectors, which may be positive or negative, whole, fractional or incommensurable, but which is always *real*, is called a **Scalar**, because it may be always found by the actual comparison of the parallel vectors with a parallel right line as a scale.

It is to be observed that tensors are pure numbers, or signless numbers, operating only *metrically* on the lengths of the vectors of which they are coefficients: while scalars are sign-bearing numbers, or the *reals* of Algebra, and are combined with each other by the ordinary rules of Algebra; they may be regarded as the product of tensors and the signs of direction.

Thus, let

$$\alpha = a\mathrm{U}\alpha.$$

Then $\mathrm{T}\alpha = a$. If we increase the length of α by the factor b, b is a tensor, but the tensor of the resulting vector is ba. If we operate with $-b$, $-b$ is not a tensor, for α is not only stretched but also reversed; the tensor of the resulting vector is as before ba; in other words, direction does not enter into the conception of a tensor. As the product of a sign and a tensor, $-b$ is a scalar. The operation of taking the scalar terms of an expression is indicated by the symbol S. Thus, if c be any real algebraic quantity,

$$\mathrm{S}\,(-ba\mathrm{U}\alpha + c) = c,$$

for $-ba\mathrm{U}\alpha$ is a vector, and the only scalar term in the expression is c.

9. It is evident from Art. 7 that if a, b, c are scalar coefficients, and α any vector, we have

$$(a + b + c)\,\alpha = a\alpha + b\alpha + c\alpha \quad . \quad . \quad . \quad . \quad (6).$$

Furthermore, if (Fig. 6)

$$\mathrm{OA} = \alpha, \quad \mathrm{AB} = \beta, \quad \mathrm{BC} = \gamma, \quad \mathrm{OA}' = m\alpha,$$

then, $A'B'$ being drawn parallel to AB and $B'C'$ to BC,

$$A'B' = m\beta, \quad B'C' = m\gamma.$$

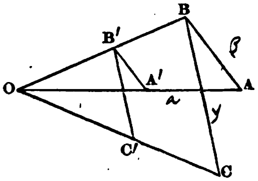
Now

$$OC = \alpha + \beta + \gamma,$$

and

$$OC' = mOC = m(\alpha + \beta + \gamma).$$

Fig. 6.



But we have also

$$\begin{aligned} OC' &= OA' + A'B' + B'C' \\ &= m\alpha + m\beta + m\gamma. \end{aligned}$$

Hence

$$m(\alpha + \beta + \gamma) = m\alpha + m\beta + m\gamma \quad . \quad . \quad . \quad (7),$$

or *the distributive law holds good for the multiplication of scalar and vector quantities.*

10. It is clear that while

$$\alpha - \alpha = 0,$$

$\alpha \pm \beta$ cannot be zero, since no amount of transference in a direction not parallel to α can affect α.

Hence, if

$$n\alpha + m\beta = 0,$$

since α and β are entirely independent of each other, we must have

$$n\alpha = 0 \quad \text{and} \quad m\beta = 0,$$

or

$$n = 0 \quad \text{and} \quad m = 0.$$

Or, if

$$m\alpha + n\beta = m'\alpha + n'\beta,$$

then

$$m = m' \quad \text{and} \quad n = n'.$$

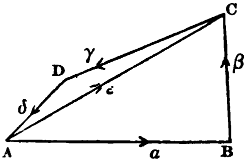
And, in general, if

then

$$\left.\begin{aligned} \Sigma\alpha + \Sigma\beta = 0, \\ \Sigma\alpha = 0 \quad \text{and} \quad \Sigma\beta = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8).$$

Three or more vectors may, however, neutralize each other. Thus (Fig. 7)

Fig. 7.



$$\alpha+\beta+\gamma+\delta=0,$$
$$\epsilon-\beta-\alpha=0,$$

and this whether ABCD be plane or gauche. In any closed figure, therefore, we have

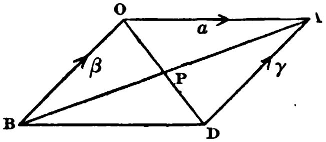
$$\alpha+\beta+\gamma+\delta+\cdots\cdots=0,$$

where α, β, γ, δ, $\cdots\cdots$, are the vector sides in order.

11. Examples.

1. *The right lines joining the extremities of equal and parallel right lines are equal and parallel.*

Fig. 8.



Let OA and BD (Fig. 8) be the given lines, and $\mathrm{OA}=\alpha$, $\mathrm{BO}=\beta$, $\mathrm{DA}=\gamma$. Then, by condition, $\mathrm{BD}=\alpha$.

Now,

$$\mathrm{BA}=\mathrm{BO}+\mathrm{OA}=\beta+\alpha\,;$$

also,

$$\mathrm{BA}=\mathrm{BD}+\mathrm{DA}=\alpha+\gamma\,;$$

or, equating the values of BA,

$$\beta+\alpha=\alpha+\gamma.$$

Hence (Art. 2), $\gamma=\beta$, and BO is parallel and equal to DA.

2. *The diagonals of a parallelogram bisect each other.*

In Fig. 8 we have

$$\mathrm{BD}=\mathrm{OA}=\mathrm{OP}+\mathrm{PA}\,;$$

also

$$\mathrm{BD}=\mathrm{BP}+\mathrm{PD}\,;$$

$$\therefore\ \mathrm{OP}+\mathrm{PA}=\mathrm{BP}+\mathrm{PD}.$$

But, OP and PD being in the same right line,

$$\mathrm{OP}=m\mathrm{PD}.$$

Similarly

$$\mathrm{PA}=n\mathrm{BP}.$$

Hence

$$m\mathrm{PD} + n\mathrm{BP} = \mathrm{PD} + \mathrm{BP},$$

$$m = 1, \quad n = 1,$$

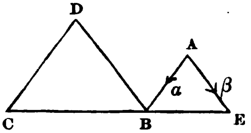
and

$$\mathrm{OP} = \mathrm{PD}, \quad \mathrm{BP} = \mathrm{PA}.$$

3. *If two triangles, having two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel, the remaining sides shall be in a straight line.*

Let (Fig. 9) $\mathrm{AB} = \alpha$, $\mathrm{AE} = \beta$. Then, by condition, $\mathrm{DC} = x\alpha$, $\mathrm{DB} = x\beta$.

Fig. 9.



Now

$$\mathrm{CB} = \mathrm{CD} + \mathrm{DB} = x\ (\beta - \alpha).$$

But

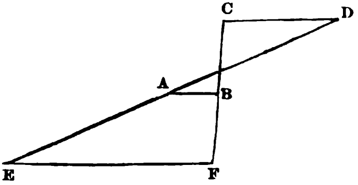
$$\mathrm{BE} = \beta - \alpha.$$

Hence (Art. 2), B being a common point, CB and BE are one and the same right line.

4. *If two right lines join the alternate extremities of two parallels, the line joining their centers is half the difference of the parallels.*

We have (Fig. 10)

Fig. 10.



$$\mathrm{AB} = \mathrm{AD} + \mathrm{DC} + \mathrm{CB},$$

and, also,

$$\mathrm{AB} = \mathrm{AE} + \mathrm{EF} + \mathrm{FB}.$$

Adding

$$2\,\mathrm{AB} = \overset{O}{(\mathrm{AD} + \mathrm{AE})} + (\mathrm{DC} + \mathrm{EF}) + \overset{O}{(\mathrm{CB} + \mathrm{FB})}$$

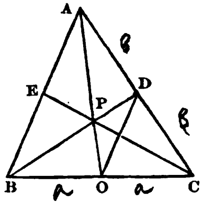
$$= \mathrm{EF} - \mathrm{CD};$$

or, as lines,

$$\mathrm{AB} = \tfrac{1}{2}\,(\mathrm{EF} - \mathrm{CD}).$$

5. *The medials of a triangle meet in a point and trisect each other.*

Fig. 11.



Let (Fig. 11) $BO = \alpha$, $CD = \beta$. Then $OC = \alpha$, $DA = \beta$.

Now

$$BA = 2\alpha + 2\beta = 2(\alpha + \beta),$$

and, since $OD = (\alpha + \beta)$, BA and OD are parallel.

Again

$$BP + PA = BA = 2\,OD = 2\,(OP + PD).$$

But BP and PD, as also OP and PA, lie in the same direction, and therefore

$$BP = 2\,PD \quad \text{and} \quad PA = 2\,OP.$$

Hence the medials OA and DB trisect each other.

Draw CP and PE. Then

$$BP = 2\,PD = \tfrac{2}{3}BD = \tfrac{2}{3}(2\alpha + \beta),$$

and

$$CP = CB + BP = \tfrac{2}{3}(2\alpha + \beta) - 2\alpha = \tfrac{2}{3}(\beta - \alpha),$$

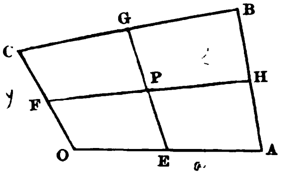
$BE = \frac{1}{2} BA =$ $$PE = PB + BE = \alpha + \beta - \tfrac{2}{3}(2\alpha + \beta) = \tfrac{1}{3}(\beta - \alpha).$$

Hence PE and CP are in the same straight line, or the medials meet in a point.

6. *In any quadrilateral, plane or gauche, the bisectors of opposite sides bisect each other.*

We will first find a value for OP (Fig. 12) under the supposition that P is the middle point of GE. We shall then find a value for OP, under the supposition that P is the middle point of FH. If these expressions prove to be identical, these middle points must coincide. In this, as in many other problems, the solution depends upon reaching the same point by different routes and comparing the results.

Fig. 12.



Let $\mathrm{OA} = \alpha$, $\mathrm{OB} = \beta$, $\mathrm{OC} = \gamma$.

1st.
$$\mathrm{OC} + \mathrm{CG} = \mathrm{OE} + \mathrm{EG}. \qquad (a)$$
But
$$\mathrm{CG} = \tfrac{1}{2}\mathrm{CB} = \tfrac{1}{2}(\beta - \gamma),$$
which, in (a), gives
$$\gamma + \tfrac{1}{2}(\beta - \gamma) = \tfrac{1}{2}\alpha + \mathrm{EG}.$$

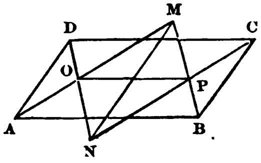
by subtr.
$$\therefore\ \mathrm{EP} = \tfrac{1}{2}\mathrm{EG} = \tfrac{1}{4}(\gamma + \beta - \alpha),$$
$$\begin{aligned} \mathrm{OP} = \mathrm{OE} + \mathrm{EP} &= \tfrac{1}{2}\alpha + \tfrac{1}{4}(\gamma + \beta - \alpha) \\ &= \tfrac{1}{4}(\alpha + \beta + \gamma). \qquad (b) \end{aligned}$$

2d.
$$\mathrm{FH} - \tfrac{1}{2}\mathrm{AB} = \mathrm{FO} + \mathrm{OA},$$
or
$$\mathrm{FH} - \tfrac{1}{2}(\beta - \alpha) = -\tfrac{1}{2}\gamma + \alpha.$$
$$\therefore\ \mathrm{FP} = \tfrac{1}{2}\mathrm{FH} = \tfrac{1}{4}(\alpha + \beta - \gamma),$$
$$\begin{aligned} \mathrm{OP} = \mathrm{OF} + \mathrm{FP} &= \tfrac{1}{2}\gamma + \tfrac{1}{4}(\alpha + \beta - \gamma) \\ &= \tfrac{1}{4}(\alpha + \beta + \gamma), \end{aligned}$$

which is identical with (b). Hence, the middle points of FH and GE coincide.

7. *If* ABCD (Fig. 13) *be any parallelogram, and* OP *any line parallel to* DC, *and the indicated lines be drawn, then will* MN *be parallel to* AD.

Fig. 13.

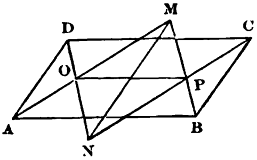


Let $\mathrm{AM} = \alpha$, $\mathrm{BM} = \beta$.
Then
$$\mathrm{AO} = r\alpha,$$
$$\mathrm{AD} = s\alpha + q\beta,$$
$$\mathrm{OD} = -r\alpha + s\alpha + q\beta.$$

We have
$$\mathrm{NM} = \mathrm{NO} + \mathrm{OM} = \mathrm{NP} + \mathrm{PM},$$
in which
$$\mathrm{NO} = x(-r\alpha + s\alpha + q\beta),$$
$$\mathrm{OM} = (1 - r)\alpha,$$
$$\mathrm{NP} = y(-r\beta + s\alpha + q\beta),$$
$$\mathrm{PM} = (1 - r)\beta.$$

Substituting in the above equation, we obtain, by Art. 10,

Fig. 13.



$$-xr + xs + 1 - r = ys,$$
$$xq = -yr + yq + 1 - r.$$

Eliminating y

$$x = \frac{1-r}{r}.$$

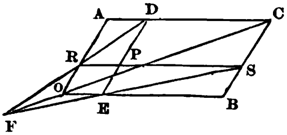
Substituting this value in

$$\begin{aligned}\text{NM} &= \text{NO} + \text{OM} \\ &= \frac{1-r}{r}(-r\alpha + s\alpha + q\beta) + (1-r)\,\alpha \\ &= \frac{1-r}{r}(s\alpha + q\beta) \\ &= \frac{1-r}{r}\,\text{AD}.\end{aligned}$$

Hence AD and NM are parallel.

8. *If, through any point in a parallelogram, lines be drawn parallel to the sides, the diagonals of two of the parallelograms so formed will intersect on the diagonal of the original parallelogram.*

Fig. 14.



Let (Fig. 14) $\text{OA} = \alpha$, $\text{OB} = \beta$. Then $\text{OR} = m\alpha$, $\text{OE} = n\beta$.

We have

$$\text{RD} = \text{RO} + \text{OE} + \text{ED} = n\beta + (1-m)\,\alpha,$$
$$\text{ES} = \text{EO} + \text{OR} + \text{RS} = m\alpha + (1-n)\,\beta.$$

Also

$$\text{FO} = \text{FR} + \text{RO} = x\text{RD} + \text{RO} = x\,[n\beta + (1-m)\,\alpha] - m\alpha, \quad (a)$$

and

$$\text{FO} = \text{FE} + \text{EO} = y\text{ES} + \text{EO} = y\,[m\alpha + (1-n)\,\beta] - n\beta. \quad (b)$$

From (*a*) and (*b*)

$$nx = y\,(1-n) - n \quad \text{and} \quad x\,(1-m) - m = ym.$$

Eliminating y

$$x = \frac{m}{1-m-n}.$$

Substituting this value of x in (a)

$$\mathrm{FO} = \frac{m}{1-m-n}\left[n\beta + (1-m)\alpha\right] - m\alpha$$

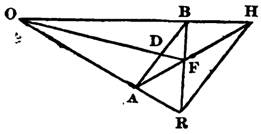
$$= \frac{mn}{1-m-n}(\beta+\alpha),$$

or, FO and $\mathrm{OC} = (\beta + \alpha)$ are in the same straight line.

9. *If, in any triangle* OAB (Fig. 15), *a line* OD *be drawn to the middle point of* AB, *and be produced to any point, as* F, *and the sides of the triangle be produced to meet* AF *and* BF *in* H *and* R, *then will* HR *be parallel to* AB.

Let $\mathrm{OA} = \alpha$, $\mathrm{OB} = \beta$. Then $\mathrm{OR} = x\alpha$, $\mathrm{OH} = y\beta$, $\mathrm{AB} = \beta - \alpha$.

Fig. 15.



Now

$$\mathrm{OD} = \mathrm{OA} + \tfrac{1}{2}\mathrm{AB} = \tfrac{1}{2}(\alpha+\beta).$$

Also, $\mathrm{OF} = z(\alpha + \beta)$, that is, some multiple of OD.

Then, 1st.

$$\mathrm{BR} = p\,\mathrm{BF},$$

$$-\beta + x\alpha = p(-\beta + \mathrm{OF})$$

$$= p[-\beta + z(\alpha+\beta)];$$

$$\therefore\ x = pz \quad\text{and}\quad -1 = pz - p. \qquad (a)$$

Eliminating z

$$p = x + 1.$$

And, 2d.

$$\mathrm{AH} = q\,\mathrm{AF},$$

$$-\alpha + y\beta = q(-\alpha + \mathrm{OF})$$

$$= q[-\alpha + z(\alpha+\beta)];$$

$$\therefore\ y = qz \quad\text{and}\quad -1 = qz - q. \qquad (b)$$

Eliminating z

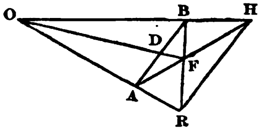
$$q = y + 1.$$

From (a) and (b)

$$z = \frac{x}{p} = \frac{y}{q},$$

and, since $p = x + 1$ and $q = y + 1$,

Fig. 15.



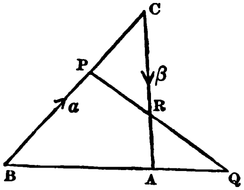
$$x = y \quad \text{and} \quad p = q.$$

$$\therefore \text{RH} = \text{RO} + \text{OH} = y\beta - x\alpha = x(\beta - \alpha)$$
$$= x\text{AB},$$

or, RH and AB are parallel.

10. *If any line* PR (Fig. 16) *be drawn, cutting the two sides of any triangle* ABC, *and be produced to meet the third side in* Q, *then*

Fig. 16.



$$\text{PC} \cdot \text{BQ} \cdot \text{RA} = \text{CR} \cdot \text{AQ} \cdot \text{BP}.$$

Let $\text{BP} = \alpha$, $\text{CR} = \beta$. Then $\text{PC} = p\alpha$, $\text{RA} = r\beta$ and $\text{BA} = \text{BC} + \text{CA} = (1 + p)\alpha + (1 + r)\beta$.

We have

$$\text{AQ} = x\text{BA} = x\left[(1 + p)\alpha + (1 + r)\beta\right],$$

as also

$$\text{AQ} = \text{AR} + \text{RQ} = -r\beta + y\text{PR} = -r\beta + y(p\alpha + \beta).$$
$$\therefore x(1 + p) = yp \quad \text{and} \quad x(1 + r) = -r + y.$$

Eliminating y

$$x = (1 + x)pr;$$

whence

$$\frac{\text{AQ}}{\text{BA}} = \frac{\text{BQ}}{\text{BA}} \cdot \frac{\text{PC}}{\text{BP}} \cdot \frac{\text{RA}}{\text{CR}},$$

or

$$\text{PC} \cdot \text{BQ} \cdot \text{RA} = \text{CR} \cdot \text{AQ} \cdot \text{BP}.$$

11. *If triangles are equiangular, the sides about the equal angles are proportional.*

Let (Fig. 17) $\text{BC} = \alpha$, $\text{CA} = \beta$. Then $\text{BE} = m\alpha$, $\text{ED} = n\beta$, $\text{BD} = m\alpha + n\beta$ and $\text{BA} = \alpha + \beta$.

Now

$$\text{BD} = p\text{BA},$$
$$m\alpha + n\beta = p(\alpha + \beta).$$

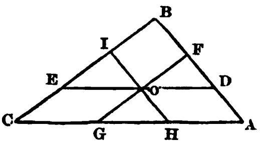
Whence

$$m = p, \quad n = p \quad \text{and} \quad m = n.$$
$$\therefore \text{BE} : \text{BC} :: \text{ED} : \text{CA}.$$

12. *If, through any point* O (Fig. 17), *within a triangle* ABC, *lines be drawn parallel to the sides, then will*

$$\frac{\text{ED}}{\text{CA}}+\frac{\text{GF}}{\text{CB}}+\frac{\text{HI}}{\text{AB}}=2.$$

Fig. 17.



Let CA $=\beta$, CB $=\alpha$. Then AB $=\alpha-\beta$, ED $=m\beta$, HI $=p\,(\alpha-\beta)$ and GF $=n\alpha$.

We have

$$\text{CO}=\text{CG}+\text{GO}=\text{CH}+\text{HO}. \qquad (a)$$

Now, as lines,

$$\frac{\text{GF}}{\text{CB}}=\frac{\text{GA}}{\text{CA}}=n, \qquad \therefore\ \text{CG}=\text{CA}-\text{GA}=(1-n)\,\beta.$$

$$\frac{\text{EB}}{\text{CB}}=\frac{\text{ED}}{\text{CA}}=m, \qquad \therefore\ \text{GO}=\text{CE}=\text{CB}-\text{EB}=(1-m)\,\alpha.$$

$$\frac{\text{DB}}{\text{AB}}=\frac{\text{DE}}{\text{AC}}=m, \qquad \therefore\ \text{HO}=\text{AD}=\text{AB}-\text{DB}=(1-m)\,(\alpha-\beta).$$

Substituting in (*a*)

$$(1-n)\,\beta+(1-m)\,\alpha=p\beta+(1-m)(\alpha-\beta),$$

or (Art. 10)

$$n+m+p=2.$$

12. Complanar *vectors are those which lie in, or parallel to, the same plane.* If α, β, γ are any vectors in space, they are complanar when equal vectors, drawn from a common origin, lie in the same plane.

If α, β, γ are complanar, but not parallel, a triangle can always be constructed, having its sides parallel to and some multiple of α, β, γ, as $a\alpha$, $b\beta$, $c\gamma$. If we go round the sides of the triangle in order, we have

$$a\alpha+b\beta+c\gamma=0.$$

If α, β, γ are not complanar, conceive a plane parallel to two of them, as α and β. In this plane two lines may be drawn parallel to and some multiple of α and β, as $a\alpha$ and $b\beta$; and these two vectors may be represented by $p\delta$ (Art. 3).

Now $p\delta$, being in the same plane with $a\alpha$ and $b\beta$, cannot therefore be equal to γ, or to any multiple of it; $p\delta$ and γ cannot therefore (Art. 10) neutralize each other. Hence

$$p\delta + c\gamma = a\alpha + b\beta + c\gamma \qquad \text{cannot be zero.}$$

If, then, we have the relation

$$a\alpha + b\beta + c\gamma = 0$$

between non-parallel vectors, they are complanar; or, if α, β, γ be not complanar, and the above relation be true, then, also,

$$a = 0, \quad b = 0, \quad c = 0.$$

13. Co-initial *vectors are those which denote transference from the same point.*

(*a*). *If three co-initial vectors are complanar, and give the relations,*

$$\left.\begin{aligned} &(a) \quad a\alpha + b\beta + c\gamma = 0 \\ &(b) \quad a + b + c = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (9),$$

they will terminate in a straight line.

For, let OA $= \alpha$ (Fig. 15), OB $= \beta$, OD $= \gamma$. Then DA $= \alpha - \gamma$, BA $= \alpha - \beta$.

From Equation (9), (*b*)

$$(a + b + c)\,\alpha = 0,$$

from which, subtracting (*a*) of Equation (9),

$$b\,(\alpha - \beta) + c\,(\alpha - \gamma) = 0,$$
$$b\text{BA} + c\text{DA} = 0\,;$$

and, since these two vectors neutralize each other, and have a common point, they are on the same straight line. Hence, A, D and B are in the same straight line.

(*b*). *Conversely, if α, β, γ are co-initial, complanar and terminate in the same straight line, and a, b, c have such values as to render*

$$a\alpha + b\beta + c\gamma = 0,$$

then will

$$a + b + c = 0.$$

For

$$\text{DA} = \alpha - \gamma \quad \text{and} \quad \text{BA} = \alpha - \beta.$$

But, by condition,

$$\alpha - \beta = x\,(\alpha - \gamma),$$

or

$$(1 - x)\,\alpha - \beta + x\gamma = 0,$$

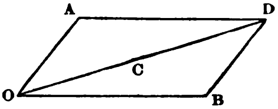
in which

$$(1 - x) - 1 + x = 0.$$

14. Examples.

1. *The extremities of the adjacent sides of a parallelogram and the middle point of the diagonal between them lie in the same straight line.*

Fig. 18.



Let OA $= \alpha$, OB $= \beta$, OC $= \gamma$.
Then

$$\text{OD} = \text{OB} + \text{BD},$$
$$2\gamma - \beta - \alpha = 0.$$

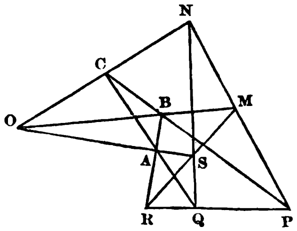
But, also,

$$2 - 1 - 1 = 0,$$

hence, B, C and A are in the same straight line (Art. 13).

2. *If two triangles,* ABC *and* SMN (Fig. 19), *are so situated that lines joining corresponding angles meet in a point, as* O, *then the pairs of corresponding sides produced will meet in three points,* P, Q, R, *which lie in the same straight line.*

Fig. 19.



Let OA $= \alpha$, OB $= \beta$, OC $= \gamma$.
Then OS $= m\alpha$, OM $= n\beta$,
ON $= p\gamma$, BA $= \alpha - \beta$,
MS $= m\alpha - n\beta$,
BR $= x\,(\alpha - \beta)$ and
MR $= y\,(m\alpha - n\beta)$.

1st. BM = BR − MR,
or

$$n\beta - \beta = x\,(\alpha - \beta) - y\,(m\alpha - n\beta),$$
$$\therefore\ n - 1 = -x + yn,\quad x - my = 0.$$

Eliminating y

$$x = -\frac{m\,(n-1)}{m-n}.$$

Also

$$\text{OR} = \text{OB} + \text{BR} = \beta + x\ (\alpha - \beta) = \beta - \frac{m\ (n-1)}{m-n}\ (\alpha - \beta),$$

whence

$$\text{OR} = \frac{n\ (m-1)\ \beta - m\ (n-1)\ \alpha}{m-n}. \qquad (a)$$

2d.

$$\text{CN} = \text{CP} - \text{NP},$$

or

$$p\gamma - \gamma = v\ (\beta - \gamma) - w\ (n\beta - p\gamma).$$

$$\therefore\ p - 1 = -v + wp, \qquad v - wn = 0.$$

Eliminating w

$$v = -\frac{n\ (p-1)}{n-p}.$$

Also

$$\text{OP} = \text{OC} + \text{CP} = \gamma + v\ (\beta - \gamma) = \gamma - \frac{n\ (p-1)}{n-p}\ (\beta - \gamma),$$

whence

$$\text{OP} = \frac{p\ (n-1)\ \gamma - n\ (p-1)\ \beta}{n-p}. \qquad (b)$$

3d. In the same manner, we obtain

$$\text{OQ} = \frac{m\ (p-1)\ \alpha - p\ (m-1)\ \gamma}{p-m}. \qquad (c)$$

From (a), (b) and (c) we observe that, clearing of fractions, and multiplying (a) by $p-1$, (b) by $m-1$, (c) by $n-1$, and adding the three resulting equations, member by member, the collected coefficients of α, β, γ, in the second member of the final equation, are separately equal to zero. Hence the first member

$$\text{OR}\,(m-n)\,(p-1) + \text{OP}\,(n-p)\,(m-1) + \text{OQ}\,(p-m)\,(n-1) = 0.$$

But

$$(m-n)\ (p-1) + (n-p)\ (m-1) + (p-m)\ (n-1) = 0.$$

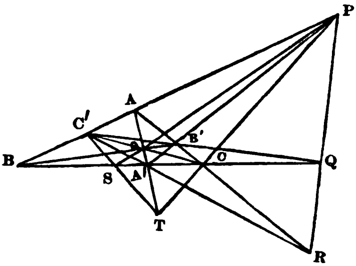
Hence, R, P and Q are in the same straight line.

3. Given the relation

$$a\alpha + b\beta + c\gamma = 0.$$

Then α, β, γ are complanar; but, if co-initial (as they may be made to be, since a vector is not changed by motion parallel to itself, *i.e.* by translation without rotation), and $a + b + c$ is not zero, they do not terminate in a straight line. Hence, if O is the origin, and A, B, C, their terminal points, A, B and C are not collinear. Let these points be joined, forming the triangle ABC (Fig. 20), and OA, OB, OC prolonged to meet the sides in A′, B′, C′. To find the relation between the segments of the sides, let

Fig. 20.



$$\text{OA}' = \alpha' = x\alpha, \qquad \text{OB}' = \beta' = y\beta, \qquad \text{OC}' = \gamma' = z\gamma,$$

whence

$$\alpha = \frac{\alpha'}{x}, \qquad \beta = \frac{\beta'}{y}, \qquad \gamma = \frac{\gamma'}{z}.$$

Substituting these in succession in the given relation,

$$\frac{a}{x}\alpha' + b\beta + c\gamma = 0,$$

$$a\alpha + \frac{b}{y}\beta' + c\gamma = 0,$$

$$a\alpha + b\beta + \frac{c}{z}\gamma' = 0,$$

whence, since A′, C, B are to be collinear,

$$\frac{a}{x} + b + c = 0,$$

and, for a like reason,

$$a + \frac{b}{y} + c = 0,$$

$$a + b + \frac{c}{z} = 0.$$

Whence

$$x = -\frac{a}{b+c}, \qquad y = -\frac{b}{a+c}, \qquad z = -\frac{c}{a+b},$$

and

$$\alpha' = -\frac{a}{b+c}\alpha, \qquad \beta' = -\frac{b}{a+c}\beta, \qquad \gamma' = -\frac{c}{a+b}\gamma,$$

or, from the given relation,

$$\alpha' = \frac{b\beta + c\gamma}{b+c}, \qquad \beta' = \frac{c\gamma + a\alpha}{c+a}, \qquad \gamma' = \frac{a\alpha + b\beta}{a+b}.$$

Whence

$$\begin{aligned} b\ (\alpha' - \beta) &= c\ (\gamma - \alpha'), \\ c\ (\beta' - \gamma) &= a\ (\alpha - \beta'), \\ a\ (\gamma' - \alpha) &= b\ (\beta - \gamma'), \end{aligned}$$

and

$$\frac{\mathrm{BA}'}{\mathrm{A}'\mathrm{C}} = \frac{c}{b}, \qquad \frac{\mathrm{CB}'}{\mathrm{B}'\mathrm{A}} = \frac{a}{c}, \qquad \frac{\mathrm{AC}'}{\mathrm{C}'\mathrm{B}} = \frac{b}{a},$$

or, multiplying,

$$\mathrm{BA}' \,.\, \mathrm{CB}' \,.\, \mathrm{AC}' \,. = \mathrm{A}'\mathrm{C} \,.\, \mathrm{B}'\mathrm{A} \,.\, \mathrm{C}'\mathrm{B}.$$

4. *If* O (Fig. 20) *be any point, and* ABC *any triangle, the transversals through* O *and the vertices divide the sides into segments having the relation*

$$\mathrm{BA}' \,.\, \mathrm{CB}' \,.\, \mathrm{AC}' \,. = \mathrm{A}'\mathrm{C} \,.\, \mathrm{B}'\mathrm{A} \,.\, \mathrm{C}'\mathrm{B}.$$

Let $\mathrm{A}'\mathrm{C} = \alpha$, $\mathrm{BC} = a\alpha$, $\mathrm{CB}' = \beta$, $\mathrm{CA} = b\beta$. Then $\mathrm{BA} = a\alpha + b\beta$.

Also let

$$\mathrm{BO} = x\mathrm{BB}', \qquad \mathrm{OA} = y\mathrm{A}'\mathrm{A}, \qquad \mathrm{BC}' = m\mathrm{BA}, \qquad \mathrm{CC}' = z\mathrm{CO}.$$

Then

$$\begin{aligned}
\mathrm{BO} &= x\mathrm{BB}' = x(\mathrm{BC}+\mathrm{CB}') = x(a\alpha+\beta),\\
\mathrm{OA} &= y\mathrm{A}'\mathrm{A} = y(\mathrm{A}'\mathrm{C}+\mathrm{CA}) = y(\alpha+b\beta),\\
\mathrm{BC}' &= m\mathrm{BA} = m(a\alpha+b\beta),\\
\mathrm{CC}' &= z\mathrm{CO} = z(\mathrm{CB}+\mathrm{BO}) = z[-a\alpha+x(a\alpha+\beta)].
\end{aligned}$$

From the triangle BOA we have

$$\mathrm{BO}+\mathrm{OA}+\mathrm{AB}=0,$$

$$x(a\alpha+\beta)+y(\alpha+b\beta)-b\beta-a\alpha=0.$$

$$\therefore\quad xa+y-a=0,\qquad x+yb-b=0.$$

Eliminating y $$x=\frac{b(1-a)}{1-ba}.$$

From the triangle BCC'

$$\mathrm{BC}+\mathrm{CC}'+\mathrm{C}'\mathrm{B}=0,$$

$$a\alpha+z[-a\alpha+x(a\alpha+\beta)]-m(a\alpha+b\beta)=0,$$

whence, as usual, and substituting the above value of x,

$$1-m=z-z\frac{b(1-a)}{1-ba},\qquad m=z\frac{1-a}{1-ba},$$

or

$$\frac{1-m}{m}=\frac{1-b}{1-a}.$$

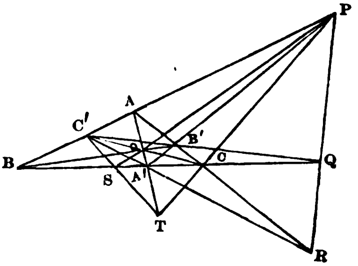
Substituting m, b and a,

$$\frac{\mathrm{C}'\mathrm{A}}{\mathrm{BC}'}=\frac{\mathrm{AB}'}{\mathrm{B}'\mathrm{C}}\cdot\frac{\mathrm{CA}'}{\mathrm{A}'\mathrm{B}},$$

which is the required relation.

5. *If* (Fig. 20) *lines be drawn through* A', B', C', *and produced to meet the opposite sides of the triangle in* P, Q, R, *then are* P, Q *and* R *collinear.*

Fig. 20.



With the notation of the last example,

$$\mathrm{BC}' = m\mathrm{BA} = \frac{a-1}{a+b-2}(a\alpha + b\beta).$$

1st. From the triangle $\mathrm{C'BA'}$

$$\begin{aligned}\mathrm{C'A'} &= \mathrm{C'B} + \mathrm{BA'}\\ &= -\frac{a-1}{a+b-2}(a\alpha + b\beta) + (a-1)\alpha\\ &= \frac{a-1}{a+b-2}[(b-2)\alpha - b\beta].\end{aligned}$$

Also

$$\mathrm{A'R} = x\mathrm{C'A'} = \mathrm{A'C} + \mathrm{CR} = \mathrm{A'C} - y\beta,$$

$$x\frac{a-1}{a+b-2}[(b-2)\alpha - b\beta] = \alpha - y\beta,$$

$$\therefore\ y = \frac{b}{b-2},$$

and

$$\mathrm{BR} = \mathrm{BC} + \mathrm{CR} = a\alpha - \frac{b}{b-2}\beta. \qquad (a)$$

2d. From the triangle $\mathrm{C'AB'}$

$$\begin{aligned}\mathrm{C'B'} &= \mathrm{C'A} + \mathrm{AB'}\\ &= (1-m)(a\alpha + b\beta) + (1-b)\beta\\ &= \frac{b-1}{a+b-2}[a\alpha - (a-2)\beta].\end{aligned}$$

Also

$$\mathrm{B'Q} = x\mathrm{C'B'} = \mathrm{B'C} + \mathrm{CQ} = \mathrm{B'C} + y\alpha,$$

$$x\frac{b-1}{a+b-2}[a\alpha - (a-2)\beta] = -\beta + y\alpha,$$

$$\therefore\ y = \frac{a}{a-2},$$

and

$$\mathrm{BQ} = \mathrm{BC} + \mathrm{CQ} = (a+y)\alpha = \frac{a(a-1)}{a-2}\alpha. \qquad (b)$$

3d.

$$\mathrm{A'P} = x\mathrm{A'B'} = x(\alpha + \beta),$$

$$\mathrm{A'P} = \mathrm{A'B} + \mathrm{BP} = (1-a)\alpha + y(a\alpha + b\beta),$$

$$\therefore\ y = \frac{a-1}{a-b},$$

and

$$\mathrm{BP} = y\mathrm{BA} = \frac{a-1}{a-b}(a\alpha + b\beta). \qquad (c)$$

Multiplying the second members of (a), (b), (c), by $(a-1)(b-2)$, $-(a-2)(b-1)$, $(a-b)$ respectively, their sum is zero. Hence

$$(a-1)(b-2)\,\mathrm{BR} - (a-2)(b-1)\,\mathrm{BQ} + (a-b)\,\mathrm{BP} = 0.$$

But

$$(a-1)(b-2) - (a-2)(b-1) + (a-b) = 0.$$

Hence R, Q and P are collinear.

6. *If* PC (Fig. 20) *and* PO *be produced to meet* AA′ *and* BC, *then* T *and* S *are collinear with* C′. A similar proposition would obtain for Q and R.

With the following notation,

$$\mathrm{BA} = \alpha, \qquad \mathrm{BA}' = \beta, \qquad \mathrm{BB}' = a\alpha + b\beta,$$

we have

$$\mathrm{BO} = \mathrm{BA} + \mathrm{AB}' + \mathrm{B}'\mathrm{O} = \mathrm{BA}' + \mathrm{A}'\mathrm{O},$$

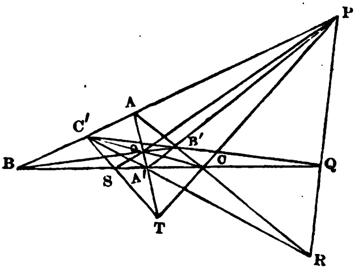
$$\alpha + b\beta - (1-a)\,\alpha + x(a\alpha + b\beta) = \beta + y(\alpha - \beta).$$

$$\therefore\ y = \frac{a}{a+b},$$

$$\mathrm{BO} = \frac{b\beta + a\alpha}{a+b};$$

also

Fig. 20.



$$\mathrm{BP} = \mathrm{BA}' + \mathrm{A}'\mathrm{P} = \mathrm{BA} + \mathrm{AP};$$

$$\beta + z[a\alpha + (b-1)\beta] = \alpha + w\alpha,$$

$$\therefore\ w = \frac{a-1+b}{1-b},$$

$$\mathrm{BP} = \frac{a\alpha}{1-b};$$

and

$$\mathrm{BC} = \mathrm{BA}' + \mathrm{A}'\mathrm{C} = \mathrm{BA} + \mathrm{AC},$$

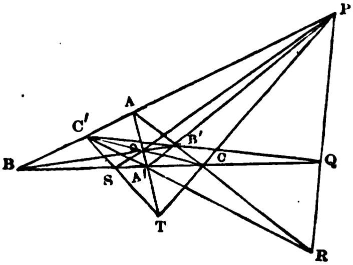
$$\beta + v\beta = \alpha + u[(1-a)\,\alpha - b\beta],$$

$$\therefore\ v=\frac{a+b-1}{1-a},$$

$$\mathrm{BC}=\frac{b\beta}{1-a}.$$

Now to find BS, BC′ and BT, we have

Fig. 20.



1st.

$$\mathrm{BS}=x'\mathrm{BA}'=\mathrm{BP}+y'\mathrm{PO},$$

$$\therefore\ x'=-\frac{b}{1-2b-a},$$

$$\mathrm{BS}=-\frac{b\beta}{1-2b-a}.$$

2d.

$$\mathrm{BC}'=v'\mathrm{BA}=\mathrm{BC}+u\mathrm{CO},$$

$$\therefore\ v'=\frac{a}{2a+b-1},$$

$$\mathrm{BC}'=\frac{a\alpha}{2a+b-1}.$$

3d.

$$\mathrm{BT}=\mathrm{BA}'+\mathrm{A'T}=\mathrm{BA}'+z'\mathrm{A'O}=\mathrm{BP}+w'\mathrm{PC},$$

$$\therefore\ z'=\frac{a+b}{a-b},$$

$$\mathrm{BT}=\frac{b\beta-a\alpha}{b-a}.$$

Clearing of fractions and adding

$$(1-2b-a)\,\mathrm{BS}+(2a+b-1)\,\mathrm{BC}'+(b-a)\,\mathrm{BT}=0,$$

as also

$$(1-2b-a)+(2a+b-1)+(b-a)=0.$$

Hence S, C′ and T are collinear.

15. A medial vector is one drawn from the origin of two co-initial vectors to the middle point of the line joining their extremities.

Thus (Fig. 21), if P is the middle point of AB, OP is a medial vector. To find an expression for it, let $OA = \alpha$, $OB = \beta$, then

$$OP = OA + AP = \alpha + AP,$$
$$OP = OB + BP = \beta - AP,$$

or, adding,

$$OP = \frac{\alpha + \beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

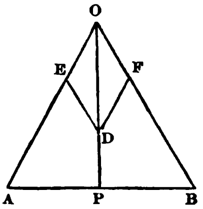
The signs in this expression will, of course, depend upon the original assumptions. Thus, if $AO = \alpha$,

$$OP = -\alpha + AP = \beta - AP,$$
$$OP = \frac{\beta - \alpha}{2}.$$

16. An **Angle-Bisector** is a line which bisects an angle. To find an expression for an angle-bisector as a vector, let $OE = \alpha$ (Fig. 21) and $OF = \beta$ be unit vectors along OA and OB. Complete the rhombus OEDF. Since the diagonal of a rhombus bisects the angle, OD is a multiple of OP. Now $OD = \alpha + \beta$, hence

$$OP = x(\alpha + \beta) \quad . \quad . \quad (11).$$

Fig. 21.



In this expression OP is of any length and x is indeterminate. If OP is limited, as by the line AB, then

$$AP = x(\alpha + \beta) - a\alpha,$$
$$AP = y AB = y(b\beta - a\alpha), \qquad (a)$$
$$\therefore\ x(\alpha + \beta) - a\alpha = y(b\beta - a\alpha),$$

or

$$x - a = -ya \quad \text{and} \quad x = yb.$$

Eliminating x

$$y = \frac{a}{a + b}.$$

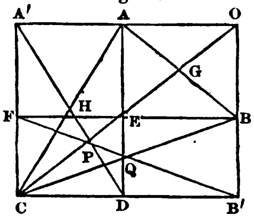
Substituting in (a)

$$AP = \frac{a}{a + b} AB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

17. Examples.

1. *If parallelograms, whose sides are parallel to two given lines, be described upon each of the sides of a triangle as diagonals, the other diagonals will intersect in a point.*

Fig. 22.



Let ABC (Fig. 22) be the given triangle. Let the diagonals B'F and A'D intersect in P, and suppose OE to meet A'D in some point as P'.

Let $\text{OA} = \alpha$, $\text{OB}' = \beta$, whence $\text{OA}' = m\alpha$, $\text{OB} = n\beta$.

Now

$$\text{B}'\text{P} - \text{DP} = \alpha. \qquad (a)$$

But

$$\text{B}'\text{P} = y\text{B}'\text{Q} = y \cdot \tfrac{1}{2}(\text{B}'\text{C} + \text{B}'\text{B}) \qquad \text{(Art. 15)}$$
$$= \tfrac{1}{2}y\,[m\alpha + (n-1)\beta].$$

And

$$\text{DP} = z\text{DH} = z \cdot \tfrac{1}{2}(\text{DC} + \text{CA}')$$
$$= \tfrac{1}{2}z\,[(m-1)\alpha - \beta].$$

Substituting in (*a*), we obtain, as usual,

$$z = \frac{2(1-n)}{1+mn-n}.$$

Again

$$\text{OP}' - \text{DP}' = \alpha + \beta. \qquad (b)$$

But

$$\text{OP}' = x\text{OG} = x \cdot \tfrac{1}{2}(\text{OA} + \text{OB})$$
$$= \tfrac{1}{2}x\,(\alpha + n\beta).$$

Substituting in (*b*) this value of OP' and $\text{DP}' = v\text{DH}$, we obtain as before,

$$v = \frac{2(1-n)}{1+mn-n}.$$

Or, $v\text{DH} = z\text{DH} = \text{DP}' = \text{DP}$. Hence, P and P' coincide, and the three diagonals meet in a point.

2. *A triangle can always be constructed whose sides are equal and parallel to the medials of any triangle.*

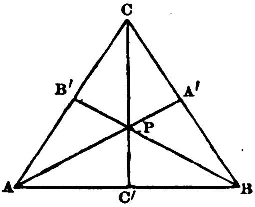
In Fig. 23 we have

$$\mathrm{AA}' = \mathrm{AB} + \mathrm{BA}' = \mathrm{AB} + \tfrac{1}{2}\mathrm{BC}.$$
$$\mathrm{BB}' = \mathrm{BC} + \tfrac{1}{2}\mathrm{CA}.$$
$$\mathrm{CC}' = \mathrm{CA} + \tfrac{1}{2}\mathrm{AB}.$$
$$\therefore\ \mathrm{AA}' + \mathrm{BB}' + \mathrm{CC}' = \tfrac{3}{2}(\mathrm{AB} + \mathrm{BC} + \mathrm{CA}) = 0. \quad \text{(Art. 10).}$$

3. *The angle-bisectors of a triangle meet in a point.*

Let α, β, γ be unit vectors along BC, AC, AB (Fig. 23).

Fig. 23.



Then (Art. 16)

$$\mathrm{AP} = x(\gamma + \beta),$$
$$\mathrm{BP} = y(\alpha - \gamma). \qquad (a)$$

Now

$$\mathrm{BC} = \mathrm{AC} - \mathrm{AB},$$
$$a\alpha = b\beta - c\gamma \qquad (b)$$

where a, b, c are the lengths of the sides.

Substituting α from (b) in (a)

$$\mathrm{BP} = y\left(\frac{b\beta - c\gamma}{a} - \gamma\right).$$

We have also

$$\mathrm{CP} = \mathrm{AP} - \mathrm{AC} = x(\gamma + \beta) - b\beta, \qquad (c)$$
$$\mathrm{CP} = \mathrm{BP} + \mathrm{CB} = y\left(\frac{b\beta - c\gamma}{a} - \gamma\right) + c\gamma - b\beta.$$
$$\therefore\ x = -\frac{yc}{a} - y + c, \quad x - b = \frac{yb}{a} - b.$$

Eliminating y

$$x = \frac{cb}{a + b + c}.$$

Substituting in (c)

$$\mathrm{CP} = \frac{cb}{a + b + c}(\gamma + \beta) - b\beta$$
$$= \frac{b}{a + b + c}[c\gamma - (a + b)\beta]$$
$$= \frac{b}{a + b + c}(-a\alpha - a\beta)$$
$$= p(\alpha + \beta).$$

Hence (Art. 16) CP is an angle-bisector.

18. *The* **Mean Point** *of any polygon is that to which the vector is the mean of the vectors to the angles.*

Hence, to find the mean point, add the vectors to the angles and divide by the number of the angles. Thus, if α_1, α_2, α_3 .... be the vectors to the angles, the vector to the mean point is

$$\alpha = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n}{n} \quad . \quad . \quad . \quad . \quad (13),$$

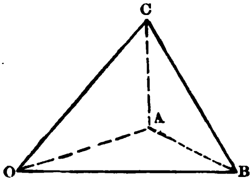
where n is the number of the angles.

The mean point of a polyedron is similarly defined. It coincides in either case, as will appear later, with the center of gravity of a system of equal particles situated at the vertices of the polygon or polyedron.

19. Examples.

1. *The mean point of a tetraedron is the mean point of the tetraedron formed by joining the mean points of the faces.*

Fig. 24.



Let (Fig. 24) OA $= \alpha$, OB $= \beta$, OC $= \gamma$. The vectors from O to the mean points of the faces are

$$\tfrac{1}{3}(\alpha + \beta + \gamma),$$
$$\tfrac{1}{3}(\alpha + \gamma),$$
$$\tfrac{1}{3}(\alpha + \beta),$$
$$\tfrac{1}{3}(\gamma + \beta),$$

and that to the mean point of the tetraedron formed by joining them is

$$\tfrac{1}{4}\left[\frac{\alpha + \beta + \gamma}{3} + \frac{\alpha + \beta}{3} + \frac{\alpha + \gamma}{3} + \frac{\gamma + \beta}{3}\right] = \tfrac{1}{4}(\alpha + \beta + \gamma),$$

which is the vector to the mean point of OABC.

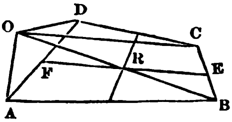
The same is true of the tetraedron formed by joining the mean points of the edges AB, BC and CA with O, since

$$\tfrac{1}{4}\left[\frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} + \frac{\alpha + \gamma}{2}\right] = \tfrac{1}{4}(\alpha + \beta + \gamma).$$

The above is, of course, independent of the origin, and would be true were O not taken at one of the vertices.

2. *The intersection of the bisectors of the sides of a quadrilateral is the mean point.*

Fig. 25.



Let (Fig. 25) $\text{OA} = \alpha$, $\text{OB} = \beta$, $\text{OC} = \gamma$, $\text{O} = \delta$, $\text{OR} = \rho$. Then (Art. 15)

$$\begin{aligned} \rho &= \tfrac{1}{2}(\text{OF} + \text{OE}) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(\alpha + \delta) + \tfrac{1}{2}(\gamma + \beta)\right] \\ &= \tfrac{1}{4}(\alpha + \beta + \gamma + \delta). \end{aligned}$$

If O is at A, then $\text{OA} = \alpha = 0$, and

$$\rho = \tfrac{1}{4}(\beta + \gamma + \delta).$$

3. *If the sides (in order) of a quadrilateral be divided proportionately, and a new quadrilateral formed by joining the points of division, then will both quadrilaterals have the same mean point.*

Let α, β, γ, δ be the vectors to the vertices of the given quadrilateral, from any initial point O.

Then, for the vector to the mean point, we have

$$\tfrac{1}{4}(\alpha + \beta + \gamma + \delta).$$

If m be the given ratio, and α', β', γ', δ' the vectors to the vertices of the second quadrilateral, then

$$\begin{aligned} \alpha' &= \alpha + m(\beta - \alpha) = (1 - m)\alpha + m\beta, \\ \beta' &= (1 - m)\beta + m\gamma, \\ \gamma' &= (1 - m)\gamma + m\delta, \\ \delta' &= \alpha + (1 - m)(\delta - \alpha) = \delta - m(\delta - \alpha); \end{aligned}$$

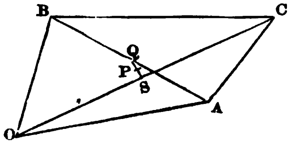
whence

$$\tfrac{1}{4}(\beta' + \gamma' + \delta' + \alpha') = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta).$$

4. *In any quadrilateral, plane or gauche, the middle point of the bisector of the diagonals is the mean point.*

Let (Fig. 26) $\mathrm{OA} = \alpha$, $\mathrm{OB} = \beta$, $\mathrm{OC} = \gamma$, $\mathrm{OS} = \frac{1}{2}\gamma$.

Fig. 26.

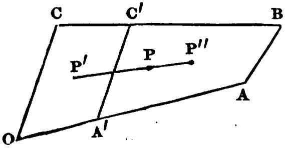


Then (Art. 15)

$$\begin{aligned}\mathrm{OP} &= \tfrac{1}{2}(\mathrm{OQ} + \mathrm{OS}) \\ &= \tfrac{1}{2}[\tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}\gamma] \\ &= \tfrac{1}{4}(\alpha + \beta + \gamma).\end{aligned}$$

5. *If the two opposite sides of a quadrilateral be divided proportionately, and the points of division joined, the mean points of the three quadrilaterals will lie in the same straight line.*

Fig. 27.



Let C′, A′ (Fig. 27) be the points of division, and m the given ratio. Then, if $\mathrm{OA} = \alpha$, $\mathrm{BC} = \gamma$, $\mathrm{OA}' = m\alpha$, $\mathrm{C'C} = m\gamma$, $\mathrm{AB} = \beta$ and O is the initial point, the vectors to the mean points P, P′, P″ are

$$\begin{aligned}\mathrm{OP} &= \tfrac{1}{4}(3\alpha + 2\beta + \gamma), \\ \mathrm{OP}' &= \tfrac{1}{4}[(m+2)\alpha + 2\beta + (2-m)\gamma], \\ \mathrm{OP}'' &= \tfrac{1}{4}[(m+3)\alpha + 2\beta + (1-m)\gamma]; \\ \therefore\ \mathrm{PP}' &= \frac{1-m}{4}(\gamma - \alpha), \\ \mathrm{P''P} &= \frac{m}{4}(\gamma - \alpha).\end{aligned}$$

Therefore, P′, P″ P are in the same straight line.

20. Exercises.

1. The diagonals of a parallelopiped bisect each other.
2. In Fig. 58, show that BG and CH are parallel.
3. If the adjacent sides of a quadrilateral be divided proportionately, the line joining the points of division is parallel to the diagonal joining their extremities.

4. The medial to the base of an isosceles triangle is an angle-bisector.

5. In any right-angled triangle ABC (Fig. 58), the lines BK, CF, AL meet in a point.

6. Any angle-bisector of a triangle divides the opposite side into segments proportional to the other two sides.

7. The line joining the middle point of the side of any parallelogram with one of its opposite angles, and the diagonal which it intersects, trisect each other.

8. If the middle points of the sides of any quadrilateral be joined in succession, the resulting figure will be a parallelogram with the same mean point.

9. The intersections of the bisectors of the exterior angles of any triangle with the opposite sides are in the same straight line.

10. If AB be the common base of two triangles whose vertices are C and D, and lines be drawn from any point E of the base parallel to AD and AC intersecting BD and BC in F and G, then is FG parallel to DC.

CHAPTER II.

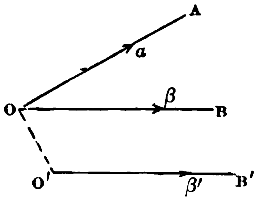
Multiplication and Division of Vectors, or Geometric Multiplication and Division.

21. Elements of a Quaternion.

The quotient of two vectors is called a **Quaternion.**

We are now to see what is meant by the quotient of two vectors, and what are its elements.

Fig. 28.



Let α and β' (Fig. 28) be two vectors drawn from O and O' respectively and not lying in the same plane; and let their quotient be designated in the usual way by $\frac{\alpha}{\beta'}$.

Whatever their relative positions, we may always conceive that one of these vectors, as β', may be moved parallel to itself so that the point O' shall move over the line O'O to O. The vectors will then lie in the same plane. Since neither the length or direction of β' has been changed during this parallel motion, we have $\beta = \beta'$, and the quotient of any two vectors, α, β', will be the same as that of two equal co-initial vectors, as α and β. We are then to determine the ratio $\frac{\alpha}{\beta}$, in which α and β lie in the same plane and have a common origin O.

Whatever the nature of this quotient, we are to regard it as some factor which *operating on the divisor produces the dividend*, *i.e.* causes β to coincide with α in direction and length, so that if this quotient be q, we shall have, by definition,

$$q\beta = \alpha \quad \text{when} \quad \frac{\alpha}{\beta} = q \; . \; . \; . \; . \; . \quad (14).$$

If at the point O′ we suppose a vector O′C $= \gamma$ to be drawn, not parallel to the plane AOB, and that this vector be moved as before, so that O′ falls at O, the plane which, after this motion, γ will determine with α, will differ from the plane of α and β, so that if the quotient

$$\frac{\alpha}{\gamma} = q',$$

q and q' will differ because their planes differ. Hence we conclude that the quotients q and q' cannot be the same if α, β and γ are not parallel to one plane, and therefore that the position of the plane of α and β must enter into our conception of the quotient q.

Again, if γ be a vector O′C, parallel to the plane AOB, but differing as a vector from β, then when moved, as before, into the plane AOB, it will make with α an angle other than BOA. Hence the angle between α and β must also enter into our conception of q. This is not only true as regards the *magnitude* of the angle, but also its *direction.* If, for example, γ have such a direction that, when moved into the plane AOB, it lies on the other side of α, so that AOC on the left of α is equal to AOB, then the quotient q', of $\frac{\alpha}{\gamma}$, in operating on γ to produce α must turn γ in a direction opposite to that in which $q = \frac{\alpha}{\beta}$ turns β to produce α. Therefore q and q' will differ unless the angles between the vector dividend and divisor are in each the same, both as regards magnitude and direction of rotation. Of the two angles through which one vector may be turned so as to coincide with the other is meant the lesser, and it will therefore, generally, be $< 180°$.

Finally, if the lengths of β and γ differ, then $\frac{\alpha}{\beta} = q$ will still differ from $\frac{\alpha}{\gamma} = q'$. Therefore the ratio of the lengths of the vectors must also enter into the conception of q.

We have thus found the quotient q, regarded as an operator which changes β into α, to depend upon the plane of the vectors, the angle between them and the ratio of their lengths. Since two angles are requisite to fix a plane, it is evident that q depends upon *four* elements, and performs *two* distinct operations:

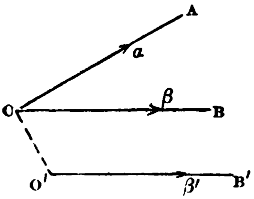
1st. A stretching (or shortening) of β, so as to make it of the same length as α;

2d. A turning of β, so as to cause it to coincide with α in direction,

the order of these two operations being a matter of indifference.

Of the four elements, the turning operation depends upon three; two angles to fix the plane of rotation, and one angle to fix the amount of rotation in that plane. The stretching operation depends only upon the remaining one, *i.e.*, upon the ratio of the vector lengths. As depending upon four elements we observe one reason for calling q a quaternion. The two operations of which q is the symbol being entirely independent of each other, a quaternion is a *complex quantity*, decomposable, as will be seen, into two factors, one of which stretches or shortens the vector divisor so that its length shall equal that of the vector dividend, and is a signless number called the *Tensor of the quaternion;* the other turns the vector divisor so that it shall coincide with the vector dividend, and is therefore called the *Versor of the quaternion.* These factors are symbolically represented by Tq and Uq, read "tensor of q" and "versor of q," and q may be written

Fig. 28.



$$q = \mathrm{T}q \cdot \mathrm{U}q.$$

22. An *equality between two quaternions* may be defined directly from the foregoing considerations.

If the plane of α and β be moved parallel to itself; or if the angle AOB (Fig. 28), remaining constant in magnitude and estimated in the same direction, be rotated about an axis through O perpendicular to the plane; or the absolute lengths of α and β

vary so that their ratio remains constant, q will remain the same. Hence if

$$\frac{\alpha}{\beta} = q \quad \text{and} \quad \frac{\alpha'}{\beta'} = q',$$

then will

$$q = q',$$

when

1st. *The vector lengths are in the same ratio*, and

2d. *The vectors are in the same or parallel planes*, and

3d. *The vectors make with each other the same angle both as to magnitude and direction.*

The plane of the vectors and the angle between them are called, respectively, the plane and angle of the quaternion, and the expression $\frac{\alpha}{\beta}$, a *geometric fraction* or *quotient*. It is to be observed that q has been regarded as the operator *on* β, producing α. This must be constantly borne in mind, for it will subsequently appear that if we write $q\beta = \alpha$ to express the operation by which q converts β into α, $q\beta$ and βq will not in general be equal.

23. Since q, in operating upon β to produce α, must not only turn β through a definite angle but also in a definite direction, some convention defining positive and negative rotation with reference to an axis is necessary.

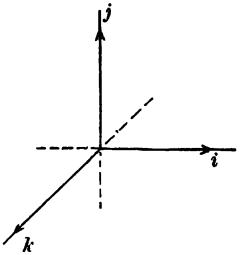
By *positive* rotation with reference to an axis is meant *left-handed* rotation when the direction of the axis is *from* the plane of rotation *towards* the eye of a person who stands on the axis facing the plane of rotation.

[If the direction of the axis is regarded as from the eye towards the plane of rotation, positive rotation is righthanded. Thus, in facing the dial of a watch, the motion of the hands is positive rotation relatively to an axis from the eye towards the dial. For an axis pointing from the dial to the eye, the motion of the hands is negative rotation. Or again, the rotation of the earth from west to east is negative relative to an axis from north to south, but positive relative to an axis from south to north.]

On the above assumption, if a person stand on the axis, facing the positive direction of rotation, the positive direction of the axis will always be from the place where he stands towards the *left*.

If i, k, j (Fig. 31) be three axes at right angles to each other, with directions as indicated in the figure, then positive rotation is from i to j, from j to k, and from k to i, relatively to the axes k, i, j respectively. A precisely opposite assumption would be equally proper. The above is in accordance with the usual method of estimating positive angles in Trigonometry and Mechanics.

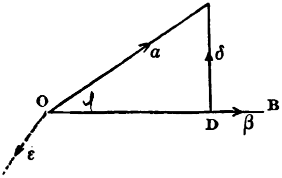
Fig. 31 (*bis*).



24. Let OA and OB (Fig. 29) be any two co-initial vectors whose lengths are a and b, α and β being unit vectors along OA and OB, so that

$$\text{OA} = a\alpha,$$
$$\text{OB} = b\beta.$$

Fig. 29.



Let the angle AOB between the vectors be represented by ϕ; also draw AD perpendicular to OB, and let the unit vector along DA be δ. The tensor of OD is evidently $a\cos\phi$ and that of DA $a\sin\phi$. If we assume that, as in Algebra, geometrical quotients which have a common divisor are added and subtracted by adding and subtracting the numerators over the common denominator, so that

$$\frac{\alpha}{\beta} \pm \frac{\gamma}{\beta} = \frac{\alpha \pm \gamma}{\beta}$$

then, since

$$\text{OA} = \text{OD} + \text{DA},$$

we have

$$\frac{\text{OA}}{\text{OB}} = \frac{\text{OD} + \text{DA}}{\text{OB}} = \frac{\text{OD}}{\text{OB}} + \frac{\text{DA}}{\text{OB}}$$
$$= \frac{a\cos\phi \, . \, \beta}{b \, . \, \beta} + \frac{a \sin\phi \, . \, \delta}{b \, . \, \beta}$$
$$= \frac{a}{b}\left(\frac{\cos\phi \, . \, \beta}{\beta} + \frac{\sin\phi \, . \, \delta}{\beta}\right).$$

a [illegible] = no. [illegible]
β = unit [illegible]

We have already defined (Art. 8) the quotient of two parallel vectors as a scalar, and in the first term of the parenthesis, β being a unit vector, $\frac{\beta}{\beta} = 1$, and

$$\frac{\text{OA}}{\text{OB}} = \frac{a}{b}\left(\cos\phi + \sin\phi \cdot \frac{\delta}{\beta}\right). \qquad (a)$$

The last term contains the quotient $\frac{\delta}{\beta}$ of two unit vectors at right angles to each other. This quotient is to be regarded, as before, as a factor which, operating on the divisor β, produces δ, *i.e.*, turns β left-handed through an angle of 90°; and this quotient must designate the plane of rotation and the direction of rotation. If we define the effect of any unit vector, operating as a multiplier upon another at right angles to it, to be the turning of the latter in a positive direction through an angle of 90° in a plane perpendicular to the operator, then the unit vector ϵ, drawn from O perpendicular to the plane of δ and β, and in the direction indicated in the figure, will be the factor which operating on β produces δ, and

$$\epsilon\beta = \delta \quad \text{or} \quad \frac{\delta}{\beta} = \epsilon.$$

The unit vector ϵ, as an axis, indicates the plane of rotation; its direction determines the direction of rotation, and by definition its rotating effect extends through an angle of 90°; as a quotient, therefore, it completely determines the operator which changes β into δ. Equation (a) thus becomes

$$\frac{\text{OA}}{\text{OB}} = \frac{a}{b}(\cos\phi + \epsilon\sin\phi),$$

or, if OA and OB be themselves denoted by α and β, and the tensors of α and β by $T\alpha$ and $T\beta$,

$$q = \frac{T\alpha}{T\beta}(\cos\phi + \epsilon\sin\phi) \quad . \quad . \quad . \quad . \quad (15),$$

in which $\frac{T\alpha}{T\beta}$ is the tensor of q, being the ratio of the vector lengths, and $\cos\phi + \epsilon \sin\phi$ is the versor of q, its plane, determined by the axis ϵ, and angle ϕ being the plane and angle of the quaternion.

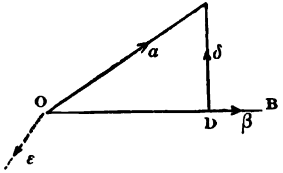
When α and β are of the same length, or $T\alpha = T\beta$, $Tq = \frac{T\alpha}{T\beta} = 1$, and the effect of q as a factor, or operator, is simply one of version.

Like **T**, the symbol **U** is one of operation, indicating the operation of taking the versor, so that

$$Uq = \cos\phi + \epsilon \sin\phi.$$

This operation takes into account but one of the two distinct acts which we have seen the quotient q must perform, as an agent converting β into α, namely, the act of version; it thus eliminates the quantitative element of length. In this respect it is similar to the reduction of a vector to its unit of length, an operation which also eliminates this same element of length, and has been designated by the same symbol **U**.

Fig. 29.



When α and β are at right angles to each other, $\phi = 90°$, and the versor $\cos\phi + \epsilon \sin\phi$ reduces to the unit vector ϵ, which has been defined, as an operator, to be a versor turning a line at right angles to it through an angle 90°. Any vector, therefore, as α, contains, in its unit vector in the same direction, a versor element or factor of which $U\alpha$ is the symbol, **U** indicating the reduction of α to its unit of length or the taking of its versor factor. Hence the appellation versor of α (Art. 7).

If in Equation (15) the vectors be reduced to the unit of length,

$$\frac{U\alpha}{U\beta} = Uq = U\frac{\alpha}{\beta}.$$

25. We may now express the relation

$$\frac{\alpha}{\beta}=\frac{\mathrm{T}\alpha}{\mathrm{T}\beta}(\cos\phi+\epsilon\sin\phi)=q \qquad \text{(Eq. 15)}$$

in the symbolic notation

or

$$\left.\begin{aligned}\frac{\alpha}{\beta}&=\mathrm{T}\frac{\alpha}{\beta}\,.\,\mathrm{U}\frac{\alpha}{\beta}\\ q&=\mathrm{T}q\,.\,\mathrm{U}q\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ .\ (16),$$

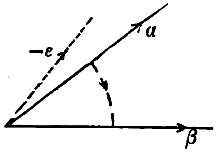
and say that *the quotient of two vectors is the product of a tensor and a versor;* and that

1st. *The tensor of the quotient,* $\left(\frac{\mathrm{T}\alpha}{\mathrm{T}\beta}\right)$, *is the ratio of their tensors;*

2d. *The versor of the quotient,* $(\cos\phi+\epsilon\sin\phi)$, *is the cosine of the contained angle plus the product of its sine and a unit vector, at right angles to their plane and such that the rotation which causes the divisor to coincide in direction with the dividend shall be positive.*

26. If, for $\frac{\alpha}{\beta}=q$, we write $\frac{\beta}{\alpha}=q'$, it is evident that q' differs from q both in the act of tension and version; the tensor of q' being the reciprocal of the tensor of q, and the unit vector ϵ, while still parallel to its former position, is reversed in direction (Art. 23) since the direction of rotation is reversed (Fig. 30). Hence

Fig. 30.

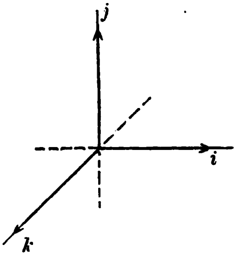


$$\frac{\beta}{\alpha}=\frac{\mathrm{T}\beta}{\mathrm{T}\alpha}(\cos\phi-\epsilon\sin\phi)\quad .\ .\ .\ .\ (17).$$

$\frac{\beta}{\alpha}$ is called the *reciprocal* of $\frac{\alpha}{\beta}$. As already remarked, the positive direction of ϵ is a matter of choice. It is only necessary that if we have $+\epsilon$ in $\mathrm{U}\frac{\alpha}{\beta}$, we must have $-\epsilon$ in $\mathrm{U}\frac{\beta}{\alpha}$, or conversely.

27. Let i, j, k (Fig. 31) represent unit vectors at right angles to each other. The effect of any unit vector acting as a multiplier upon another at right angles to it, has been defined (Art. 24) to be the turning of the latter in a positive direction in a plane perpendicular to the operator or multiplier through an angle of 90°. Thus, i operating on j produces k.

Fig. 31.



This operation is called multiplication, and the result the product, and is expressed as usual

$$ij = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The quotient of two vectors being a factor which converts the divisor into the dividend, we have also

$$\frac{k}{j} = i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

either *the product or quotient of two unit vectors at right angles to each other being a unit vector perpendicular to their plane.*

This multiplication is evidently not that of algebra; it is a revolution, which for rectangular vectors extends through 90°. Nor is k in Equation (18) a numerical product, nor i in Equation (19) a numerical quotient. This kind of multiplication and division is called *geometric.*

In accordance with the above definition we may write the following equations:

$$\left.\begin{array}{ll} ij = k & \frac{k}{j} = i \\ jk = i & \frac{i}{k} = j \\ ki = j & \frac{j}{i} = k \\ ji = -k & \frac{-k}{i} = j \\ kj = -i & \frac{-i}{j} = k \end{array}\right\} \quad . \quad . \quad . \quad . \quad (20).$$

$$\left.\begin{array}{ll}
ik=-j & \frac{-j}{k}=i \\
i(-j)=-k & \frac{-k}{-j}=i \\
i(-k)=j & \frac{j}{-k}=i \\
k(-i)=-j & \frac{-j}{-i}=k \\
k(-j)=i & \frac{i}{-j}=k \\
j(-k)=-i & \frac{-i}{-k}=j \\
j(-i)=k & \frac{k}{-i}=j
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (20).$$

Since the effect of i, k, j as operators is to turn a line from one direction into another which differs from it by 90°, they are called *quadrantal versors*.

28. Since

$$i \times j = k \quad \text{and} \quad i \times k = -j = -1 \times j,$$

we have

$$i \times i \times j = -1 \times j,$$

or

$$i \times i = -1.$$

We may denote the continued use of i as an operator by an exponent which indicates the number of times it is so used. This is consistent with the meaning of an exponent in algebraic notation. In both cases it denotes the number of times the operator is used, in one instance as a numerical factor, in the other as a versor. Thus

$$jjiii = j^2 i^3, \quad \frac{iii}{jj} = \frac{i^3}{j^2}, \quad \text{etc.}$$

In conformity to this notation the above equation becomes

$$i^2 = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

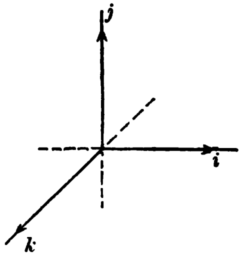
and in a similar manner,

$$\left.\begin{array}{l} j^2=-1 \\ k^2=-1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Hence *the square of a unit vector is* -1.

The meaning of the word "square" is more general than that which it possesses in Algebra, as was that of the word "product" in Art. 27. The propriety of this extension of meaning lies in the fact that for certain special cases, the processes above defined reduce to the usual algebraic processes to which these terms were originally restricted. The conclusion $i^2=-1$ is seen to follow directly from the definition, since if i operates twice in succession on either $\pm j$ or $\pm k$, it turns the vector, in either case successively through two right angles, so that after the operation it points in the opposite direction. A similar reversal would have resulted if the minus sign had been written before the vector. Thus $-(\pm j)=\mp j$. Hence $i \times i$, or i^2, as an operator, has the effect of the minus sign in reversing the direction of a line.

Fig. 31.



29. It is to be observed that so long as the *cyclical order* $i, j, k, i, j, k, i, \ldots$ is maintained, the product of any two of these three vectors gives the third; thus

$$ij=k, \quad jk=i, \quad ki=j;$$

and therefore

$$\begin{aligned} (ij)k &= kk = k^2=-1, \\ (jk)i &= ii = i^2=-1, \\ (ki)j &= jj = j^2=-1; \end{aligned}$$

as also

$$\begin{aligned} i(jk) &= ii = i^2=-1, \\ j(ki) &= jj = j^2=-1, \\ k(ij) &= kk = k^2=-1, \end{aligned}$$

hence

$$i(jk)=(ij)k,$$
$$j(ki)=(jk)i,$$
$$k(ij)=(ki)j,$$

which involves the *Associative law.*

We may therefore omit the parentheses and write

$$ijk=jki=kij=-1 \quad . \quad . \quad . \quad . \quad . \quad (23),$$

or, *the continued product of three rectangular unit vectors is the same so long as the cyclical order is maintained.*

But

$$k(ji)=k(-k)=-k^2=1 \quad . \quad . \quad . \quad . \quad (24),$$

or, *a change in the cyclical order reverses the sign of the product.*

30. In Equation (24) we have assumed that

$$k(-k)=-kk.$$

That this is the case appears from the fact that i operating on $-j$ produces $-k$, or

$$i(-j)=-k,$$

and that the same result would be obtained by operating with i on j, producing k, and then *reversing* k. That is, to turn the negative, or reverse, of a vector through a right angle, is the same as turning the vector through a right angle and then reversing it. *The negative sign is*, therefore, *commutative with* i, j, k, or

$$i(-j)=-ij=-k \quad . \quad . \quad . \quad . \quad . \quad (25).$$

31. It follows directly from the definition of multiplication, as applied to rectangular unit vectors, that the commutative property of algebraic factors does not hold good. For

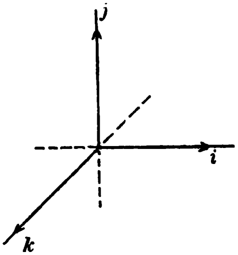
$$ij=k,$$

but

$$ji=-k.$$

Hence, *to change the order of the factors is to reverse the sign of the product.* The operator is always written first; and, since the order cannot be changed without affecting the result, in reading such an expression as $ij = k$, this sequence of the factors must be indicated by saying "*i into j* equals k" and not "i multiplied by j equals k," the latter not being true.

Fig. 31.



Hence also the conception of a quotient as a factor requires a similar distinction, which in Algebra is unnecessary. In the latter, from $\frac{c}{b} = a$ we have, indifferently, $ab = c$ and $ba = c$. But from $\frac{k}{j} = i$, while $ij = k$ is true, $ji = k$ is not true. In expressing therefore the relations between i, j and k by multiplication instead of division, care must be taken to conform to the definition, the quotient being used as the multiplier or operator *on* the divisor. This non-commutative property of rectangular unit vectors, which results directly from the primary definition of the operation of multiplication, will be seen hereafter to extend to vectors in general and to quaternions, whose multiplication is not commutative except in special cases.

The quotient then being a factor which operates on the divisor to produce the dividend, we have

$$\frac{k}{j}\,j = k, \quad \text{that is,} \quad \frac{k}{\cancel{j}}\cancel{j} = k \quad . \quad . \quad . \quad . \quad (26),$$

the cancelling being performed by an upward right-handed stroke. But $\cancel{j}\frac{k}{\cancel{j}} = k$ is not true, for this would involve $ji = ij$.

32. It follows also that the directions of rotation of a fraction, as $\frac{k}{j}$, and its reciprocal are opposite. Thus

$$\frac{k}{j} = i, \quad \frac{j}{k} = -i \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

and therefore that the reciprocal of the quotient i is $-i$, or

$$\frac{1}{i} = -i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28);$$

that is, *the reciprocal of a unit vector is the vector reversed.* This may be written

$$\frac{1}{i} = i^{-1} = -i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

the exponent denoting that, as a factor or versor, i is used once, while the minus sign before the exponent indicates a reversal in the direction of rotation.

33. If α be any unit vector, we obtain from the preceding Article

$$\alpha\frac{1}{\alpha} = \alpha(-\alpha)$$
$$= -\alpha\alpha = -\alpha^2 = 1.$$

But

$$\cancel{\frac{1}{\alpha}\alpha} = 1,$$

hence

$$\frac{1}{\alpha}\alpha = \alpha\frac{1}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

or, *a unit vector and its reciprocal are commutative and their product plus unity.*

If α is not a unit vector,

$$\alpha = \mathrm{T}\alpha\mathrm{U}\alpha,$$
$$\frac{1}{\alpha} = \frac{1}{\mathrm{T}\alpha\mathrm{U}\alpha} = -\frac{1}{\mathrm{T}\alpha}\mathrm{U}\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

the tensor of the reciprocal of a vector being the reciprocal of its tensor.

It must be carefully observed that a fraction, as $\frac{k}{i}$, cannot be written indifferently $k\frac{1}{i}$ or $\frac{1}{i}k$, for this would involve $ki^{-1} = i^{-1}k$, which is not true.

By definition $k(-i) = -j$, or $ki^{-1} = k\frac{1}{i} = -j = \frac{k}{i}$. Hence, $\frac{k}{i} = k\frac{1}{i}$ or ki^{-1}. From the meaning attached to the ordinary notation of algebra,

$$i\frac{k}{i} = \frac{k}{i}i \qquad (a)$$

would appear to be correct; for, cancelling, we have $k = k$. Whereas, since $\frac{k}{i}$ must be written $k\frac{1}{i}$, we should have

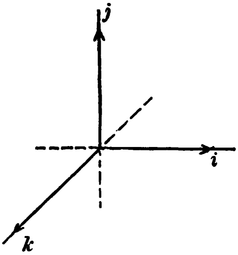
$$iki^{-1}\,[= -iki] = ki^{-1}i\,[= k]$$

or

$$ji\,[= -k] = k,$$

which is not true. Of course that equation (a) is false is directly evident from the fact that $\frac{k}{i} = -j$, and (a) involves $i(-j) = (-j)i$ or $ij = ji$. The above, however, shows that, as cancelling must be performed by an upward *right-handed* stroke when the expression is in the form of a quotient or fraction, so when expressed in the form of multiplication, the cancelled factors must be *adjacent*.

Fig. 31.



In such an expression as

$$\frac{j}{i} \cdot \frac{-i}{j} = -ji^{-1}ij^{-1} = jj = -1 \qquad (b)$$

it might be supposed permissible to write also

$$\frac{\cancel{j}}{i} \cdot \frac{-i}{\cancel{j}} = \frac{-i}{i} = -1, \qquad (c)$$

since in either case the correct result is obtained. This arises, however, from the fact that both the fractions in the first member of (b) are equal to k, and therefore may be permuted so as to read $kk = \frac{-i}{\cancel{j}} \cdot \frac{\cancel{j}}{i} = \frac{-i}{i} = -1$. The process of (c) is, however, illegitimate, and the result is correct, not because the process is so, but because the factors are in this case commutative.

34. Since the act of tension is independent of that of version, and their order is immaterial,

$$xi\,.\,yj = xy\,.\,ij = yx\,.\,ij = zk \quad . \quad . \quad . \quad (32),$$

where x and y are any two scalars and $xy = z$. Hence the commutative principle applies to tensors. If then α, β, γ are in the direction of i, j and k respectively, and a, b, c are their tensors,

$$\begin{aligned} \alpha\beta &= \mathrm{T}\alpha\mathrm{T}\beta\,.\,ij = ab\,.\,k, \\ \alpha\gamma &= \mathrm{T}\alpha\mathrm{T}\gamma\,.\,ik = -ac\,.\,j, \text{ etc.,} \end{aligned}$$

or, the product of any two rectangular vectors is the product of their tensors and a unit vector at right angles to their plane.

So also

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{\mathrm{T}\alpha\,.\,i}{\mathrm{T}\beta\,.\,j} = \frac{\mathrm{T}\alpha}{\mathrm{T}\beta}\,.\,\frac{i}{j} = -\frac{a}{b}k, \\ \frac{\alpha}{\gamma} &= \frac{\mathrm{T}\alpha\,.\,i}{\mathrm{T}\gamma\,.\,k} = \frac{a}{c}j, \text{ etc.,} \end{aligned}$$

or, the quotient of two rectangular vectors is the quotient of their tensors times a unit vector at right angles to their plane.

35. If, as above, $\alpha = ai$, then

$$\begin{aligned} \alpha\alpha &= ai\,.\,ai, \\ \alpha^2 &= a^2 i^2, \\ \alpha^2 &= -a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33). \end{aligned}$$

Hence, *the square of any vector is minus the square of its tensor.* Since $\mathrm{T}\alpha = a$ is the ratio of the lengths of α and $\mathrm{U}\alpha$, *the square of any vector is the square of the corresponding line*, regarded as a length or distance only, *with its sign changed.*

If $ai = \alpha$ and $bi = \beta$,

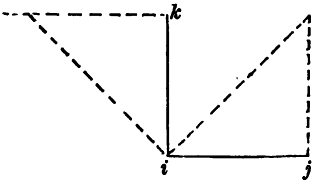
$$\alpha\beta = abi^2 = -ab.$$

36. That the multiplication of rectangular vectors is a *distributive* operation may be seen directly from Fig. 32 by observing that

$$i(j+k)=ij+ik \quad (34),$$

i being perpendicular to and in front of the plane of the paper.

Fig. 32.



37. Exercises in the transformations of i, j, k:

1. $j(-i)=k.$
2. $j(-k)=$
3. $k(-j)=i.$
4. $k(-i)=$
5. $-k(j)=i.$
6. $(-k)i=$
7. $(-j)k=-i.$
8. $(-j)(-k)=$
9. $(-j)(-i)=$
10. $(-i)(-j)=$
11. $\frac{j}{-i}=-k.$
12. $\frac{-j}{i}=$
13. $\frac{-k}{j}=$
14. $\frac{-j}{k}=$
15. $\frac{-j}{-k}=$
16. $\frac{k^2}{j}=j.$
17. $\frac{j^3}{i}=$
18. $\frac{k^4}{j}=$
19. $\frac{ik}{j}=-1.$
20. $\frac{k}{ij}=$
21. $\frac{k}{ji}=$
22. $\frac{1}{j}\cdot\frac{k}{i}=$
23. $\frac{ijk}{kji}=$
24. $\frac{i}{k}\cdot\frac{j}{j}\cdot\frac{k}{i}=$

25. Is it correct to write, in general, the product of any fractions, as $\frac{k}{j}\cdot\frac{i}{j}$, in the form $\frac{ki}{jj}$?

26. State whether $\frac{-i}{k}\cdot\frac{j}{i}=\frac{-ij}{ki}$ is correct or not, and why.

27. $i^2j^2k^2=-(ijk)^2.$

38. Resuming Equation (15),

$$q = \frac{\alpha}{\beta} = \frac{T\alpha}{T\beta}(\cos\phi + \epsilon\sin\phi);$$

the quaternion q was shown (Art. 25) to be the product of a tensor and a versor. It may also be regarded as the sum of two parts, the first of which $\left[\frac{T\alpha}{T\beta}\cos\phi\right]$ is a scalar, whose sign is that of the cosine of the angle (ϕ) between the vectors, while the second $\left[\frac{T\alpha}{T\beta}\sin\phi \cdot \epsilon\right]$ is a vector at right angles to their plane, whose sign depends upon the direction of rotation of the fraction $\frac{\alpha}{\beta}$. This may be expressed symbolically in the notation

$$q = \frac{\alpha}{\beta} = S\frac{\alpha}{\beta} + V\frac{\alpha}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

so that we have both

$$q = TqUq$$

and

$$q = Sq + Vq.$$

The second member of this last equation is read "scalar of q plus vector of q," Sq and Vq being respectively symbols for the scalar and vector parts of the quaternion. As already explained in the case of the symbol S, V is a symbol of operation, denoting the operation of taking the vector terms of the expression before which it is written.

The quotient of two vectors is, therefore, *the sum of a scalar and a vector.*

The scalar of the quotient $\left[Sq = \frac{T\alpha}{T\beta}\cos\phi\right]$ *is the ratio of the tensors times the cosine of the contained angle. The tensor of the vector part* $\left[TVq = \frac{T\alpha}{T\beta}\sin\phi\right]$ *is the ratio of the tensors times the sine of the contained angle. The versor of the vector part* $[UVq = \epsilon]$ *is a unit vector perpendicular to their plane, having a* *direction such that the direction of rotation of the divisor is positive or left-handed.*

Letting a and b be the tensors of α and β, and collecting the preceding expressions for facility of reference, we have

$$\left.\begin{aligned}
\mathbf{T}q &= \frac{a}{b} \\
\mathbf{U}q &= \cos\phi + \epsilon\sin\phi \\
\mathbf{S}q &= \frac{a}{b}\cos\phi \\
\mathbf{V}q &= \frac{a}{b}\sin\phi \,.\, \epsilon \\
\mathbf{TV}q &= \frac{a}{b}\sin\phi \\
\mathbf{UV}q &= \epsilon \\
\mathbf{SU}q &= \cos\phi \\
\mathbf{VU}q &= \sin\phi \,.\, \epsilon \\
\mathbf{TVU}q &= \sin\phi
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (36).$$

These expressions require no further explanation than that derived from a simple inspection of Equation (15) in connection with the meaning already assigned to **T**, **U**, **S** and **V** as symbols of operation.

39. De Moivre's Formula.

The following considerations will explain why the parenthesis $(\cos\phi + \epsilon\sin\phi)$ as a versor turns β left-handed through an angle ϕ. They also contain the quaternion interpretation of imaginary quantities.

Let $v = \sin\phi$ and $z = \cos\phi$.

Differentiating,

$$dv = \cos\phi\; d\phi, \quad dz = -\sin\phi\; d\phi,$$

or

$$dv = z d\phi, \qquad (a)$$

$$dz = -v d\phi. \qquad (b)$$

Multiplying (*a*) by $\sqrt{-1}$, and adding the result to (*b*),

$$dz + dv \cdot \sqrt{-1} = (-v + z\sqrt{-1})\, d\phi,$$

or

$$dz + dv \cdot \sqrt{-1} = (v\sqrt{-1} + z)\sqrt{-1}\; d\phi,$$

whence

$$\frac{d\,(z + v\sqrt{-1})}{z + v\sqrt{-1}} = d\phi \cdot \sqrt{-1}, \qquad (c)$$

which may be written

$$z + v\sqrt{-1} = e^{\phi\sqrt{-1}},$$

or

$$\cos\phi + \sin\phi \cdot \sqrt{-1} = e^{\phi\sqrt{-1}}, \qquad (d)$$

whence

$$\cos m\phi + \sin m\phi \cdot \sqrt{-1} = e^{m\phi\sqrt{-1}}. \qquad (e)$$

But we have from (*d*)

$$(\cos\phi + \sin\phi \cdot \sqrt{-1})^m = e^{m\phi\sqrt{-1}}, \qquad (f)$$

and therefore, from (*e*) and (*f*),

$$(\cos\phi + \sin\phi \cdot \sqrt{-1})^m = \cos m\phi + \sin m\phi \cdot \sqrt{-1} \quad (37),$$

which is the well-known formula of De Moivre.

This formula may be made the basis of a system of analytical trigonometry. Thus, for example, to deduce the formulae for the sine and cosine of the sum of two angles, we have from (*d*)

$$\cos\phi + \sin\phi\sqrt{-1} = e^{\phi\sqrt{-1}},$$
$$\cos\theta + \sin\theta\sqrt{-1} = e^{\theta\sqrt{-1}}.$$

Multiplying member by member,

$$\cos\phi\,\cos\theta + \cos\phi\,\sin\theta \cdot \sqrt{-1} + \cos\theta\,\sin\phi \cdot \sqrt{-1} - \sin\phi\,\sin\theta = e^{(\phi+\theta)\sqrt{-1}}. \qquad (g)$$

But from De Moivre's formula

$$\cos(\phi + \theta) + \sin(\phi + \theta)\sqrt{-1} = e^{(\phi+\theta)\sqrt{-1}}. \qquad (h)$$

Equating the first members of (g) and (h), since in any equation between real and imaginary quantities these are separately equal in the two members, we have

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi.$$
$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

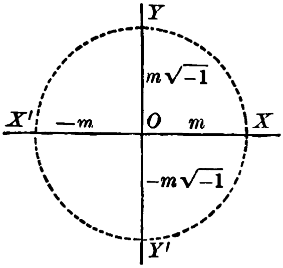
These formulae, while they may be of course demonstrated independently of De Moivre's formula, are here deduced from imaginary expressions. It would therefore appear that these expressions admit of a logical interpretation.

If any positive quantity m be multiplied by $(\sqrt{-1})^2$ the result is $-m$. That is, in accordance with the geometrical interpretation of the minus sign, we may regard the above factor $(\sqrt{-1})^2$ as having turned the linear representative of m about the origin through an angle of 180°. If, instead of multiplying m by $(\sqrt{-1})^2$, we multiply it by $\sqrt{-1}$, we may infer from analogy that the line m has been turned through an angle of 90° about the origin. If, too, we observe that each of the four expressions

$$m,\quad m\sqrt{-1},\quad -m,\quad -m\sqrt{-1}$$

is obtained from the preceding by multiplying by the factor $\sqrt{-1}$, they may be regarded as denoting in order a distance m *on* the co-ordinate axes OX, OY, OX', OY' (Fig. 33), $\sqrt{-1}$ being, as a factor, a versor turning a line left-handed through a quadrant. These expressions therefore locate a point *on* the axes, both as to distance and direction from the origin.

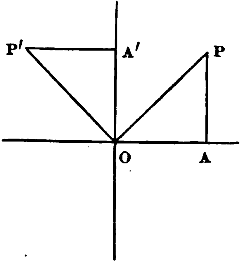
Fig. 33.



Since every imaginary expression can be reduced to the form $\pm a \pm b\sqrt{-1}$, we may, in accordance with the above interpretation of $\sqrt{-1}$, regard such an expression as defining the position of a point *out* of the axes. Thus OA $= a$ (Fig. 34) and

AP $= b$, laid off at A at right angles to OA since b is multiplied by $\sqrt{-1}$; so that in passing over OA and AP in succession we reach the point P. It is also evident that such an expression implicitly fixes the position of P by polar co-ordinates, since $\sqrt{a^2+b^2} =$ OP and tan POA $= \frac{b}{a}$. In like manner $-b + a\sqrt{-1}$ would locate a point P′, OA′ having a length $= a$, but laid off perpendicular to OA, since $\sqrt{-1}$ is a factor, and A′P′ $= -b$. As before, we have implicitly OP′ $= \sqrt{a^2+b^2}$ and tan P′OA $= -\frac{a}{b}$.

Fig. 34.



Furthermore, if we operate on the first expression, $a + b\sqrt{-1}$, which fixes the point P, with $\sqrt{-1}$, we obtain the second, $-b + a\sqrt{-1}$, or $\sqrt{-1}$ as a factor turns OP through 90° so as to make it coincide with OP′. As an operator, therefore, we may regard $\sqrt{-1}$, like i, j, k, as a quadrantal versor, turning a line through a quadrant in a positive direction. Algebraically it denotes an impossible operation. (In Algebra quantities are laid off on the same line in two opposite directions, + and −. It was because quantities are so estimated only in Algebra that Sir W. Hamilton called it the Science of Pure Time, since time can be estimated only into the future or the past.) But it is unreal or imaginary only in an algebraic sense. If the restrictions imposed by Algebra are removed, by enlarging our idea of quantity and at the same time modifying the operations to which it is subjected, this imaginary character disappears. In applying the old nomenclature to these new modifications, it will be seen that the principle of permanence is observed, *i.e.*, the new meaning of terms is an extension of the old; and when the new complex quantities reduce to those of Algebra, the new operations become identical with the old.

If now we operate upon

$$a + b\sqrt{-1},$$

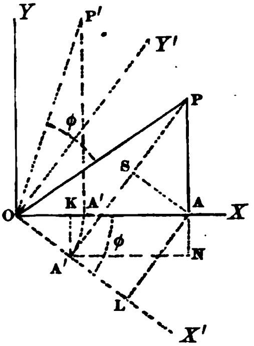
which, if we regard $a =$ OA (Fig. 35) and $b\sqrt{-1} =$ AP as vectors, is equivalent to OP, with the expression

$$\cos\phi + \sin\phi \,.\, \sqrt{-1}$$

of De Moivre's formula, we obtain

$$a\cos\phi - b\sin\phi + \sqrt{-1}\,(a\sin\phi + b\cos\phi).$$

Fig. 35.



Draw OX' so that $X'OX = \phi$; also PA″ and AL perpendicular and AS parallel to OX'. Then

$$a\cos\phi - b\sin\phi = \text{OL} - \text{A}''\text{L} = \text{OA}'',$$
$$a\sin\phi + b\cos\phi = \text{LA} + \text{SP} = \text{A}''\text{P}.$$

Make OA′ = OA″ and lay off A′P′ = A″P perpendicular to OX, since it has $\sqrt{-1}$ as a factor; then

$$(a\cos\phi - b\sin\phi) + \sqrt{-1}\,(a\sin\phi + b\cos\phi) = \text{OA}' + \text{A}'\text{P}' = \text{OP}',$$

and P′OP $= \phi$.

But the formulae for passing from a set of rectangular axes OX, OY, to another rectangular set OX', OY', are

$$x = x'\cos\phi + y'\sin\phi,$$
$$y = y'\cos\phi - x'\sin\phi,$$

in which $XOX' = \phi$, $x =$ OA, $y =$ AP, $x' =$ OA″, $y' =$ PA″ or

$$\text{OA} = \text{OK} + \text{KA},$$
$$\text{AP} = \text{NP} - \text{A}''\text{K},$$

A″K being perpendicular and A″N parallel to OX.

Hence the effect of the operator has been to turn OP left-handed through an angle ϕ, which is equivalent to turning the axes right-handed through the same angle.

$+1, -1$ and $\sqrt{-1}$ are particular cases of the general versor

$$\cos\phi + \sin\phi \,.\, \sqrt{-1},$$

namely, when ϕ is 0°, 180° and 90° respectively, $+1$ preserving, -1 reversing and $\sqrt{-1}$ semi-inverting the line operated upon.

We may now see the meaning of De Moivre's formula

$$(\cos\phi + \sin\phi \,.\, \sqrt{-1})^m = \cos m\phi + \sin m\phi \,.\, \sqrt{-1}.$$

As operators, the first member turns a line through an angle ϕ *successively* m times, while the second member turns it m times this angle *once*, producing the same result. The expressions $\cos\phi + \sin\phi \,.\, \sqrt{-1}$ and $\cos\phi + \sin\phi \,.\, \epsilon$ are identical, except that in the latter the plane of rotation is not indeterminate, being perpendicular to ϵ, $\sqrt{-1}$ *being any unit vector with indeterminate direction in space.*

Equation (37) may be put under the form

$$\cos m\,(2\pi n + \phi) + \sin m\,(2\pi n + \phi) \,.\, \sqrt{-1} = [\cos(2\pi n + \phi) + \sin(2\pi n + \phi) \,.\, \sqrt{-1}]^m.$$

In the second member if $\phi = 0$ and $m = \frac{1}{3}$, we have $\sqrt[3]{1}$ for all integral values of n, while the first member for $n = 0$, $n = 1$, $n = 2$ becomes 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}$, the three roots of unity.

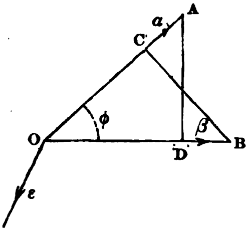
In the same way for $m = \frac{1}{6}$,

$$\sqrt[6]{1} = \begin{cases} 1, \\ \frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}, \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}, \\ -1, \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}, \\ \frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}, \end{cases}$$

the six roots of unity. The real roots lie on the axis, along which direction is assumed plus and minus, while the imaginary roots are vectors in a direction not that of the axis, and are the sum of two vectors, one of which is in the direction of the axis and the other perpendicular to it.

40. Let α and β be unit vectors along OA and OB (Fig. 36). Resolve OA $= \alpha$ into the two vectors OD, DA. Then

Fig. 36.



$$\text{OA} = \alpha = \text{OD} + \text{DA}.$$

But

$$\text{OD} = \cos\phi \cdot \beta,$$
$$\text{DA} = \epsilon(\sin\phi \cdot \beta) = \sin\phi \cdot \epsilon\beta,$$

ϵ being a unit vector perpendicular to the plane AOB, as in the figure. Hence

$$\alpha = \cos\phi \cdot \beta + \sin\phi \cdot \epsilon\beta. \qquad (a)$$

Now when α and β are unit vectors, we have by definition $\frac{\alpha}{\beta} \cdot \beta = (\cos\phi + \epsilon\sin\phi)\beta = \alpha$; or, comparing with (a),

$$(\cos\phi + \epsilon\sin\phi)\beta = \cos\phi \cdot \beta + \sin\phi \cdot \epsilon\beta.$$

The distributive law, therefore, *applies to the multiplication of a vector by the scalar and vector parts of a quaternion;* for if α and β are not unit vectors, the tensors, as merely numerical factors, can be introduced without affecting the versor conclusion. Resolve β into the vectors OC, CB, CB being perpendicular to OA. Then

$$\text{OB} = \beta = \text{OC} + \text{CB}.$$

But

$$\text{OC} = \cos\phi \cdot \alpha, \quad \text{CB} = -\epsilon(\sin\phi \cdot \alpha).$$

Hence

$$\cos\phi \cdot \alpha - \sin\phi \cdot \epsilon\alpha = \beta,$$

or, by the distributive principle,

$$(\cos\phi - \sin\phi \cdot \epsilon)\,\alpha = \beta.$$

Using the two members of this equation as multipliers on the corresponding members of (a)

$$(\cos\phi - \sin\phi\,.\,\epsilon)\,\alpha\alpha = \beta\,(\cos\phi\,.\,\beta + \sin\phi\,.\,\epsilon\beta),$$

or, since $\alpha^2 = -1$,

$$-\cos\phi + \epsilon\sin\phi = \beta\alpha \quad . \quad . \quad . \quad . \quad . \quad (38).$$

If α and β are not unit vectors,

$$\beta\alpha = \mathrm{T}\beta\mathrm{T}\alpha\,(-\cos\phi + \epsilon\sin\phi) \quad . \quad . \quad . \quad (39).$$

Operating with each member of (a) on β,

$$\begin{aligned}\alpha\beta &= (\cos\phi\,.\,\beta + \sin\phi\,.\,\epsilon\beta)\beta \\ &= \cos\phi\,.\,\beta^2 + \sin\phi\,.\,\epsilon\beta^2 \\ &= -\cos\phi - \epsilon\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),\end{aligned}$$

or, if α and β are not unit vectors,

$$\alpha\beta = \mathrm{T}\alpha\mathrm{T}\beta\,(-\cos\phi - \epsilon\sin\phi) \quad . \quad . \quad . \quad (41).$$

The product of any two vectors is, therefore, *a quaternion*, which, as before, may be regarded either as the sum of a scalar and a vector or the product of a tensor and a versor. In general notation

$$\alpha\beta = \mathrm{S}\alpha\beta + \mathrm{V}\alpha\beta = \mathrm{S}q + \mathrm{V}q \quad . \quad . \quad . \quad . \quad (42),$$

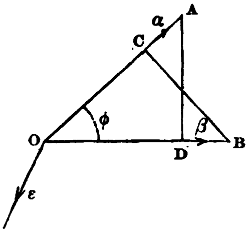
$$\alpha\beta = \mathrm{T}q\,.\,\mathrm{U}q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

The scalar of the product [$\mathrm{S}\alpha\beta = -\mathrm{T}\alpha\mathrm{T}\beta\cos\phi$] *is the product of the tensors and the cosine of the supplement of the contained angle.*

The vector of the product [$\mathrm{V}\alpha\beta = -\mathrm{T}\alpha\mathrm{T}\beta\sin\phi\,.\,\epsilon$] *has for its tensor* [$\mathrm{TV}\alpha\beta = \mathrm{T}\alpha\mathrm{T}\beta\sin\phi$] *the product of the tensors and the sine of the contained angle, and for a versor* [$\mathrm{UV}\alpha\beta = -\epsilon$] *a unit vector at right angles to their plane such that rotation about it as an axis is positive or left-handed.*

Representing the tensors of α and β by a and b, we have, as in Art. 38, from Equation (41),

Fig. 36.



$$\left.\begin{aligned} \mathrm{T}q &= ab \\ \mathrm{U}q &= -\cos\phi - \epsilon\sin\phi \\ \mathrm{S}q &= -ab\cos\phi \\ \mathrm{V}q &= -ab\sin\phi\,.\,\epsilon \\ \mathrm{TV}q &= ab\sin\phi \\ \mathrm{UV}q &= -\epsilon \\ \mathrm{SU}q &= -\cos\phi \\ \mathrm{VU}q &= -\sin\phi\,.\,\epsilon \\ \mathrm{TVU}q &= \sin\phi \\ (\mathrm{TV} : \mathrm{S})\,q &= -\tan\phi \end{aligned}\right\} \quad (44).$$

41. Resuming the expressions for the products and quotients of α and β,

$$\beta\alpha = \mathrm{T}\beta\mathrm{T}\alpha\,(-\cos\phi + \epsilon\sin\phi), \qquad (a)$$

$$\alpha\beta = \mathrm{T}\alpha\mathrm{T}\beta\,(-\cos\phi - \epsilon\sin\phi), \qquad (b)$$

$$\frac{\beta}{\alpha} = \frac{\mathrm{T}\beta}{\mathrm{T}\alpha}\,(\cos\phi - \epsilon\sin\phi), \qquad (c)$$

$$\frac{\alpha}{\beta} = \frac{\mathrm{T}\alpha}{\mathrm{T}\beta}\,(\cos\phi + \epsilon\sin\phi), \qquad (d)$$

we observe

1st. That if α and β be interchanged the sign of the vector part is changed. It is equivalent to a reversal of the angle ϕ, and consequently a change in the direction of rotation. Hence

$$\left.\begin{aligned} \mathrm{UV}\beta\alpha &= \epsilon = -\mathrm{UV}\alpha\beta \\ \mathrm{UV}\frac{\alpha}{\beta} &= \epsilon = -\mathrm{UV}\frac{\beta}{\alpha} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Vector multiplication is not therefore in general commutative.

2d. If the vectors are unit vectors,

$$\beta\alpha = -\frac{\beta}{\alpha}, \quad \alpha\beta = -\frac{\alpha}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (46),$$

the product being expressed also by a quotient. This is of course always possible, as appears from (*a*), (*b*), (*c*) and (*d*), and the transformation may be effected thus:

$$\frac{\beta}{\alpha} = -\frac{\beta \mathrm{U}\alpha}{\mathrm{T}\alpha} = \frac{\mathrm{T}\beta}{\mathrm{T}\alpha}(\cos\phi - \epsilon\sin\phi), \qquad [\text{Eq. } (31)]$$

$$-\beta\alpha = \mathrm{T}\beta\mathrm{T}\alpha(\cos\phi - \epsilon\sin\phi);$$

or

$$\beta\alpha = \mathrm{T}\beta\mathrm{T}\alpha(-\cos\phi + \epsilon\sin\phi).$$

3d. If $\phi = 0$, then in either (*a*) and (*b*) or (*c*) and (*d*) the vector part of q becomes zero, and the quaternion degrades to a scalar. When $\phi = 0$ the vectors are parallel, and $\alpha\beta = -\mathrm{T}\alpha\mathrm{T}\beta = -ab$, as in Art. 35; also $\frac{\alpha}{\beta} = \frac{\mathrm{T}\alpha}{\mathrm{T}\beta} = \frac{a}{b}$, as in Art. 8. If at the same time α and β are unit vectors $\frac{\alpha}{\beta} = \frac{\alpha}{\alpha} = 1$ [or $= \alpha\alpha^{-1} = -\alpha^2 = 1$] and $\alpha\beta = \alpha^2 = -1$, as in Arts. 33 and 28.

If then q be any quaternion and $\mathrm{V}q = 0$, *the vectors of which q is the quotient or product are parallel.*

4th. If $\phi = 90°$ then in either (*a*) and (*b*) or (*c*) and (*d*) the scalar part of q becomes zero, and the quaternion degrades to a vector; and either the product or quotient of two rectangular vectors is therefore a vector at right angles to their plane, $\alpha\beta$ reducing to $-ab\epsilon$ and $\frac{\alpha}{\beta}$ to $\frac{a}{b}\epsilon$, as in Art. 34. If at the same time α and β are unit vectors, $\alpha\beta = -\epsilon$ and $\frac{\alpha}{\beta} = \epsilon$, as in Art. 27.

If then q be any quaternion and $\mathrm{S}q = 0$, *the vectors of which q is the quotient or product are perpendicular to each other.*

5th. If an equation involves scalars and vectors, the vector terms having been so reduced as to contain no scalar parts, then since the scalar terms are purely numerical and independent of the others, *the sums of the scalars and vectors in each member are separately equal.* Thus if

$$x + a\alpha + b\beta = d + y + a'\alpha + (b' - b'')\beta$$

then

$$\left.\begin{array}{l} \\ x = d + y \quad \text{and} \quad a\alpha + b\beta = a'\alpha + (b' - b'')\beta \end{array}\right\} . \quad (47),$$

which might also be written (Art. 38)

$$\mathbf{S}(x+a\alpha+b\beta)=\mathbf{S}[d+y+a'\alpha+(b'-b'')\beta],$$
$$\mathbf{V}(x+a\alpha+b\beta)=\mathbf{V}[d+y+a'\alpha+(b'-b'')\beta].$$

6th. $\frac{\beta}{\alpha}$ being the quotient which operates on α to produce β, we have by definition

$$\frac{\beta}{\cancel{\alpha}}\cancel{\alpha}=\beta \quad \ldots\ldots \quad (48).$$

7th. $\mathbf{TV}\alpha\beta$, or $ab\sin\phi$, *is the area of a parallelogram whose sides are equal in length to* a *and* b *and parallel to* α *and* β. $\mathbf{S}\alpha\beta$, or $-ab\cos\phi$, is *numerically* the area of a parallelogram whose sides are a and b, and angle $\overset{\wedge}{ab}$ is the complement of ϕ.

8th. Since the scalar symbol $\mathbf{S}$ indicates the operation of taking the scalar terms,

$$\mathbf{S}\alpha=0 \quad \ldots\ldots \quad (49),$$

and, for a similar reason,

$$\mathbf{V}\alpha=\alpha \quad \ldots\ldots \quad (50).$$

Again, since α^2 is a scalar,

$$\mathbf{V}(\alpha^2)=0 \quad \ldots\ldots \quad (51),$$
$$\mathbf{S}(\alpha^2)=-a^2 \quad \ldots\ldots \quad (52).$$

$\mathbf{V}(\alpha^2)$ may be written $\mathbf{V}\,.\,\alpha^2$, as also $\mathbf{S}(\alpha^2)=\mathbf{S}\,.\,\alpha^2$, but these forms must be distinguished from $(\mathbf{V}\alpha)^2$ and $(\mathbf{S}\alpha)^2$, which latter are also sometimes written $\mathbf{V}^2\alpha$ and $\mathbf{S}^2\alpha$.

9th. Comparing (a) and (b),

$$\mathbf{S}\alpha\beta=\mathbf{S}\beta\alpha \quad \ldots\ldots \quad (53),$$

and

$$\mathbf{V}\alpha\beta=-\mathbf{V}\beta\alpha \quad \ldots\ldots \quad (54).$$

Adding and subtracting (a) and (b), we have also

$$\alpha\beta+\beta\alpha=2\,\mathbf{S}\alpha\beta \quad \ldots\ldots \quad (55),$$
$$\alpha\beta-\beta\alpha=2\,\mathbf{V}\alpha\beta \quad \ldots\ldots \quad (56).$$

10th. $\alpha\beta \cdot \beta\alpha = (S\alpha\beta + V\alpha\beta)(S\alpha\beta - V\alpha\beta)$ [Eqs. (53) and (54)]

$$= (S\alpha\beta)^2 - S\alpha\beta V\alpha\beta + S\alpha\beta V\alpha\beta - (V\alpha\beta)^2.$$

Hence

$$\alpha\beta \cdot \beta\alpha = (S\alpha\beta)^2 - (V\alpha\beta)^2 \quad . \; . \; . \; . \; . \quad (57),$$

or, from Equation (44),

$$\alpha\beta \cdot \beta\alpha = (T\alpha\beta)^2 \quad . \; . \; . \; . \; . \; . \quad (58).$$

42. Powers of Vectors.

The symbol i^m, m being a positive whole number, has been seen (Art. 28) to represent a quadrantal versor used m times as an operator; the exponent denoting the number of times i is used as a quadrantal versor. By an extension of this meaning of the exponent, $i^{\frac{1}{m}}$ would naturally represent a versor which, as a factor, produces the $\frac{1}{m}$th part of a quadrantal rotation. Thus $i^{\frac{1}{3}}$ produces a rotation through one-third, and $i^{\frac{3}{5}}$ through three-fifths of a quadrant, respectively. With the additional meaning attached to the negative exponent (Art. 32), as indicating a reversal in the direction of rotation, we may in general define i^t, where i is any vector-unit and t any scalar exponent, as the representative of *a versor which would cause any right line in a plane perpendicular to i to revolve in that plane through an angle $t \times 90°$*, the direction of rotation depending upon the sign of t. Hence every such power of a unit vector is a versor, and, conversely, *every versor may be represented as such a power.*

Since the angle (ϕ) of the versor is $t \times \frac{\pi}{2}$, we have $t = \frac{2\phi}{\pi}$, and any versor

$$\cos\phi + \epsilon \sin\phi$$

may be expressed

$$\cos\phi + \epsilon\sin\phi = \epsilon^{\frac{2\phi}{\pi}} \quad . \; . \; . \; . \; . \; . \quad (59),$$

and

$$\cos\phi - \epsilon\sin\phi = \epsilon^{-\frac{2\phi}{\pi}} \quad . \; . \; . \; . \; . \quad (60),$$

the vector base being the unit vector about which rotation takes place, and the exponent the fractional part of a quadrant through which rotation occurs.

The operation of which $i^{\frac{1}{2}}$ is the agent is one-half that of which i is the agent, and therefore two operations with the former is equivalent to one with the latter; or, as in Algebra,

$$i^{\frac{1}{2}}\, i^{\frac{1}{2}} = i = i^{\frac{1}{2}+\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

or, employing the other versor form, if α, β, γ are complanar unit vectors so that

$$\frac{\alpha}{\beta} = \cos\phi + \epsilon \sin\phi = \epsilon^{\frac{2\phi}{\pi}},$$

$$\frac{\beta}{\gamma} = \cos\theta + \epsilon \sin\theta = \epsilon^{\frac{2\theta}{\pi}},$$

then since

$$\frac{\alpha}{\beta} \cdot \frac{\beta}{\gamma} = \frac{\alpha}{\gamma},$$

we have

$$\begin{aligned}(\cos\phi + \epsilon\sin\phi)(\cos\theta + \epsilon\sin\theta) &= \cos\phi\cos\theta + \epsilon^2 \sin\phi\sin\theta + \\ &\qquad \epsilon(\sin\phi\cos\theta + \cos\phi\sin\theta) \\ &= \cos(\phi+\theta) + \epsilon\sin(\phi+\theta).\end{aligned}$$

The second member is the $\mathrm{U}\frac{\alpha}{\gamma}$, its angle being $(\phi+\theta)$, and may be therefore expressed as the power of a unit vector, and written $\epsilon^{\frac{2(\phi+\theta)}{\pi}}$; this exponent is the sum of the exponents of the factors, or

$$\epsilon^{\frac{2\phi}{\pi}}\,\epsilon^{\frac{2\theta}{\pi}} = \epsilon^{\frac{2(\phi+\theta)}{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

This is evidently an abridged form of notation to which *the algebraic law of indices is applicable.*

Since $\epsilon^2 = -1$ and therefore $\epsilon^4 = 1$; if $\epsilon^t = -1$, t must be an odd multiple of 2, and if $\epsilon^t = +1$, t must be an even multiple of 2.

In either case the coefficient of π in $\phi = \frac{t}{2}\pi$ is a whole number, and $\cos\phi \pm \epsilon \sin\phi$ degrades, as above, to the scalar ± 1, since $\sin m\pi = 0$ when m is an integer.

If $\epsilon^t = \pm\epsilon$, t must be an odd number; in which case also

$m=\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc., $\cos m\pi = 0$ and the versor degrades to the vector $\pm\epsilon$.

If the vector is not a unit vector, as $xi=\rho$, to interpret the exponent, say $\rho^{\frac{1}{2}}$, so as to satisfy the formula

$$\rho^{\frac{1}{2}}\rho^{\frac{1}{2}}=\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

which is analogous to Equation (61), we must combine with the conception of rotation through half a quadrant an act of tension represented by the square root of the tensor of ρ. Thus, if $x=16$, and we write

$$\rho^{\frac{1}{2}}=(16i)^{\frac{1}{2}}=16^{\frac{1}{2}}i^{\frac{1}{2}},$$

then

$$\rho^{\frac{1}{2}}\rho^{\frac{1}{2}}=(16^{\frac{1}{2}}i^{\frac{1}{2}})\,(16^{\frac{1}{2}}i^{\frac{1}{2}})=16i=\rho,$$

or, if $x=\sqrt{8}$,

$$\rho^{\frac{1}{3}}=\sqrt[6]{8}\,.\,i^{\frac{1}{3}}=\sqrt{2}\,.\,i^{\frac{1}{3}},$$
$$\rho^{\frac{1}{3}}\rho^{\frac{1}{3}}\rho^{\frac{1}{3}}=(\sqrt{2}\,.\,i^{\frac{1}{3}})(\sqrt{2}\,.\,i^{\frac{1}{3}})(\sqrt{2}\,.\,i^{\frac{1}{3}})=\sqrt{8}\,.\,i=\rho.$$

And, in general,

$$\rho^{t}=(xi)^{t}=x^{t}\,.\,i^{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

or the tensor of the power is the power of the tensor, and the versor of the power is the power of the versor. Symbolically

$$\mathbf{T}\,.\,\rho^{t}=(\mathbf{T}\rho)^{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65),$$
$$\mathbf{U}\,.\,\rho^{t}=(\mathbf{U}\rho)^{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

Any such power (ρ^{t}), as the representative of the agent of both an act of tension and version, is therefore a quaternion, whose tensor and versor can be assigned by the above rules, and, conversely, *every quaternion can be expressed as the power of a vector*, which quaternion may degrade to either a scalar or a vector as seen in the preceding versor conclusions. Hence it follows that *the index-law of Algebra is applicable to the powers of a quaternion.*

43. Relation between the Vector and Cartesian determination of a point.

If i, j, k are three unit vectors perpendicular to each other at a common point, then the vector from this point to any point P may be written

$$\rho = xi + yj + zk \quad . \quad . \quad . \quad . \quad . \quad (67),$$

in which x, y, z are the Cartesian co-ordinates of P. If the vectors are not mutually perpendicular and are represented by α, β, γ, then

$$\rho = x\alpha + y\beta + z\gamma \quad . \quad . \quad . \quad . \quad . \quad (68),$$

in which x, y, z are the Cartesian co-ordinates of P referred to the oblique axes. So long as the vectors α, β, γ are not complanar, ρ refers to any point in space.

Since any quaternion q may be expressed as the sum of a scalar and a vector, if w be any scalar, then

$$q = w + x\alpha + y\beta + z\gamma \quad . \quad . \quad . \quad . \quad . \quad (69).$$

As composed of four terms, we observe an additional reason for calling this complex expression a quaternion.

Any vector equation

$$\rho = \sigma = a\alpha + b\beta + c\gamma,$$

involves three numerical equations, as

$$x = a, \quad y = b, \quad z = c,$$

unless the vectors are complanar; in which case we may write

$$\gamma = n\alpha + m\beta,$$

and

$$\rho = (x + zn)\,\alpha + (y + zm)\beta,$$
$$\sigma = (a + cn)\,\alpha + (b + cm)\beta,$$

which, for $\rho = \sigma$, involves but two equations

$$x + zn = a + cn, \quad y + zm = b + cm.$$

Resuming the quadrinomial form of q, when the component vectors are at right angles, we have

$$\left.\begin{aligned} q &= w + xi + yj + zk \\ \mathrm{S}q &= w \\ \mathrm{V}q &= xi + yj + zk \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (70).$$

Since $(\mathrm{TV}q)^2 = -(\mathrm{V}q)^2 = x^2 + y^2 + z^2$, we have

$$\left.\begin{aligned} \mathrm{TV}q &= \sqrt{x^2 + y^2 + z^2} \\ \mathrm{UV}q &= \frac{\mathrm{V}q}{\mathrm{TV}q} = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}\right\} \quad . \; . \; . \quad (71).$$

Also, since (Art. 41, 10th.)

$$(\mathrm{T}q)^2 = (\mathrm{S}q)^2 - (\mathrm{V}q)^2 = w^2 + x^2 + y^2 + z^2,$$

$$\left.\begin{aligned} \mathrm{T}q &= \sqrt{w^2 + x^2 + y^2 + z^2} \\ \mathrm{U}q &= \frac{q}{\mathrm{T}q} = \frac{w + xi + yj + zk}{\sqrt{w^2 + x^2 + y^2 + z^2}} \\ \mathrm{SU}q &= \frac{\mathrm{S}q}{\mathrm{T}q} = \frac{w}{\sqrt{w^2 + x^2 + y^2 + z^2}} \\ \mathrm{TVU}q &= \frac{\mathrm{TV}q}{\mathrm{T}q} = \sqrt{\frac{x^2 + y^2 + z^2}{w^2 + x^2 + y^2 + z^2}} \end{aligned}\right\} \quad . \; . \quad (72).$$

44. The plane of a quaternion has been already defined as the plane of the vectors or a plane parallel to them. The axis of a quaternion is the vector perpendicular to its plane, and its angle is that included between two co-initial vectors parallel to those of the quaternion. If this angle is 90° the quaternion is called a **Right** Quaternion. Any two quaternions having a common plane, or parallel planes, are said to be **Complanar.** If their planes intersect, they are **Diplanar.** If the planes of several quaternions intersect in, or are parallel to, a common line, they are said to be **Collinear.** It follows that the axes of collinear quaternions are complanar, being perpendicular to the common line. Complanar quaternions are always collinear, and complanar axes correspond to collinear quaternions, but the latter may of course be diplanar.

Let $\frac{O'A}{O'B}$ and $\frac{O''C}{O''D}$ be any two quaternions. If complanar, they may be made to have a common plane; and, if diplanar, their planes will intersect. In the former case let OE be any line of their common plane, or, in the latter, the line of intersection of their planes. Now, without changing the ratios of their vector lengths, the planes, or the angles of the given quaternions, two lines, OF and OG, may always be found, one in each plane, or in their common plane, such that with OE we shall have

$$\frac{O'A}{O'B} = \frac{OF}{OE} \quad \text{and} \quad \frac{O''C}{O''D} = \frac{OG}{OE};$$

and, therefore, any two quaternions, considered as geometric fractions, can be reduced to a common denominator; or, in the above case

$$\frac{O'A}{O'B} + \frac{O''C}{O''D} = \frac{OF}{OE} + \frac{OG}{OE} = \frac{OF + OG}{OE}.$$

Moreover, a line OH, in the plane AO'B, may always be found such that

$$\frac{O'A}{O'B} = \frac{OE}{OH},$$

and therefore

$$\frac{O''C}{O''D} \cdot \frac{O'A}{O'B} = \frac{OG}{OE} \cdot \frac{OE}{OH} = \frac{OG}{OH},$$

and

$$\frac{O'A}{O'B} : \frac{O''C}{O''D} = \frac{OF}{OE} : \frac{OG}{OE} = \frac{OF}{OE} \cdot \frac{OE}{OG} = \frac{OF}{OG}.$$

45. Reciprocal of a Quaternion.

The reciprocal of a *scalar* is another scalar with the same sign, so that, as in Algebra, if x be any scalar, its reciprocal is $x^{-1} = \frac{1}{x}$.

The reciprocal of a *vector* has been defined (Art. 33), so that, if α be any vector, $\frac{1}{\alpha} = \alpha^{-1} = -\frac{1}{\mathrm{T}\alpha}\mathrm{U}\alpha$.

The reciprocal of a *quaternion* has also been defined (Art. 26); thus

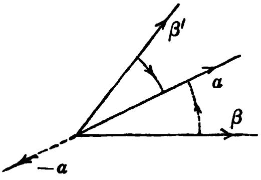
$$\frac{\alpha}{\beta} = q$$

being any quaternion,

$$\frac{\beta}{\alpha} = \frac{1}{q} = q^{-1}$$

is its reciprocal. The only difference between the quotients $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ (Fig. 37) is that, as operators, one causes β to coincide with α, while the other causes α to coincide with β. A quaternion and its reciprocal have, therefore, a common plane and equal angles as to magnitude, but opposite in direction; that is, their axes are opposite. Or

Fig. 37.



$$\angle\frac{1}{q} = \angle q \quad \text{and} \quad \text{axis}\,\frac{1}{q} = -\,\text{axis}\,q.$$

Since

$$1 : \frac{\alpha}{\beta} = \frac{\beta}{\beta} : \frac{\alpha}{\beta} = \frac{\beta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\beta}{\alpha}, \quad \text{and} \quad \frac{\beta}{\alpha} \cdot \frac{\alpha}{\beta} = \frac{\beta}{\beta} = 1,$$

the product of two reciprocal quaternions is equal to positive unity, and *each is equal to the quotient of unity by the other;* we have, therefore, as in Algebra, $\frac{1}{q}q = 1$ and $q = \frac{1}{\frac{1}{q}}$, and no new symbol is necessary for the reciprocal. $\frac{1}{q}$ is, however, sometimes written $\mathrm{R}q$, R being a general symbol of operation, namely, that of taking the reciprocal. It follows from the above that

$$\mathrm{T}\frac{1}{q} = \frac{1}{\mathrm{T}q} \quad \ldots \ldots \quad (73),$$

or, *the tensors of reciprocal quaternions are reciprocals of each other ;* while *the versors differ only in the reversal of the angle.* If then

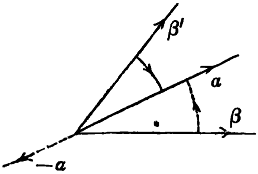
$$q = \frac{\alpha}{\beta} = \frac{\mathbf{T}\alpha}{\mathbf{T}\beta}(\cos\phi + \epsilon\sin\phi)$$

we shall have

$$\mathbf{R}q = \frac{1}{q} = q^{-1} = \frac{\beta}{\alpha} = \frac{\mathbf{T}\beta}{\mathbf{T}\alpha}(\cos\phi - \epsilon\sin\phi) \quad . \quad (74).$$

46. Conjugate of a Quaternion.

Fig. 37.



If β' (Fig. 37) be taken complanar with β and α, and making with α the same angle that β does, $\mathbf{T}\beta'$ being also equal to $\mathbf{T}\beta$, then, if $\frac{\alpha}{\beta} = q$, $\frac{\alpha}{\beta'}$ is called the **conjugate** of q, and is written $\mathbf{K}q$. The symbol **K** indicates the operation of taking the conjugate. A quaternion and its conjugate have, therefore, a common plane and tensor, as also, in the ordinary sense, equal angles; but their axes are opposite; or

$$\angle\,\mathbf{K}q = \angle\,q = \angle\,\frac{1}{q}$$

$$\mathbf{TK}q = \mathbf{T}q = \frac{1}{\mathbf{T}\frac{1}{q}}$$

and

$$\text{axis }\mathbf{K}q = -\text{ axis }q = \text{axis}\frac{1}{q} \quad . \quad . \quad . \quad (75).$$

If then

$$q = \frac{\alpha}{\beta} = \frac{\mathbf{T}\alpha}{\mathbf{T}\beta}(\cos\phi + \epsilon\sin\phi)$$

we shall have

$$\mathbf{K}q = \frac{\mathbf{T}\alpha}{\mathbf{T}\beta}(\cos\phi - \epsilon\sin\phi) \quad . \quad . \quad . \quad (76),$$

or, *the tensors of conjugate quaternions are equal*, and *the versors differ only in the reversal of the angle.*

Regarding a scalar and a vector as the limits of a quaternion

(Art. 41, 3d and 4th), we see from Equation (76) that *the conjugate of a scalar is the scalar itself*, and that

$$\mathbf{K}\alpha = -\alpha = -\mathbf{T}\alpha\mathbf{U}\alpha \quad . \quad . \quad . \quad . \quad . \quad (77),$$

or, *the conjugate of a vector is the vector reversed.* In general notation we may write

$$q = \mathbf{S}q + \mathbf{V}q,$$

whence it follows from the above that

$$\left.\begin{aligned} \mathbf{K}q &= \mathbf{S}q - \mathbf{V}q \\ \text{or (Art. 43)} \qquad \mathbf{K}q &= w - xi - yj - zk \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (78),$$

that is, *the scalar of the conjugate of a quaternion is the scalar of the quaternion*, and *the vector of the conjugate of a quaternion is the vector of the quaternion reversed ;* a result which may be expressed symbolically

$$\left.\begin{aligned} \mathbf{SK}q &= \mathbf{S}q \\ \mathbf{VK}q &= -\mathbf{V}q \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

These are Equations (53) and (54).

If we add and subtract the two conjugate quaternions

$$q = \mathbf{S}q + \mathbf{V}q, \quad \mathbf{K}q = \mathbf{S}q - \mathbf{V}q,$$

we have

$$\left.\begin{aligned} q + \mathbf{K}q &= 2\,\mathbf{S}q \\ q - \mathbf{K}q &= 2\,\mathbf{V}q \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80).$$

The sum of two conjugate quaternions is, therefore, *always a scalar*, positive or negative as the $\angle q$ is acute or obtuse. If $\angle q = \frac{\pi}{2}$, this sum is evidently zero.

Since, if q is a scalar, $\mathbf{K}q = q$, then, conversely, *if* $\mathbf{K}q = q$, *q is a scalar.*

47. Opposite Quaternions.

If, for $\frac{\alpha}{\beta}$, we write $\frac{-\alpha}{\beta}$ (Fig. 37), the latter is called the **Opposite** of q, and is evidently $-q$, for

$$\frac{-\alpha}{\beta}=\frac{0-\alpha}{\beta}=\frac{0}{\beta}-\frac{\alpha}{\beta}=0-q=-q.$$

As appears from the figure, opposite quaternions have a common plane and tensor, supplementary angles and opposite axes; or

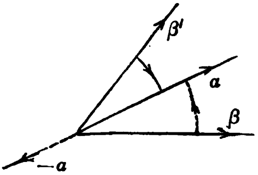
$$\mathrm{T}(-q)=\mathrm{T}q,\ \angle -q=\pi-\angle q \quad \text{and axis } (-q)=-\text{ axis } q.$$

Since
$$\frac{-\alpha}{\beta}+\frac{\alpha}{\beta}=\frac{\alpha-\alpha}{\beta}=\frac{0}{\beta}=0,$$

the sum of two opposite quaternions is zero, or

$$q+(-q)=0.$$

Fig. 37.



Also, since

$$\frac{-\alpha}{\beta}:\frac{\alpha}{\beta}=\frac{-\alpha}{\beta}\cdot\frac{\beta}{\alpha}=\frac{-\alpha}{\alpha}=-1,$$
$$\frac{-q}{q}=-1,$$

or, *their quotient is negative unity.*

If then

$$\left.\begin{aligned} q&=\frac{\alpha}{\beta}=\frac{\mathrm{T}\alpha}{\mathrm{T}\beta}(\cos\phi+\epsilon\sin\phi)\\ -q&=\frac{\mathrm{T}\alpha}{\mathrm{T}\beta}(-\cos\phi-\epsilon\sin\phi)\end{aligned}\right\}\quad\ldots\quad(81).$$

we shall have the second of these.

If $\angle q=\frac{\pi}{2}$, $\mathrm{K}q=-q$; and, conversely, if $\mathrm{K}q=-q$, q is a vector.

48. Since $\text{U}q$ is independent of the vector *lengths*, and only dependent upon relative *direction*, versors are equal whose axes and angles are the same. Hence

$$\text{UK}q = \text{U}\frac{1}{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (82).$$

But (Art. 24)

$$\text{U}\left(1 : \frac{\alpha}{\beta}\right) = \text{U}\frac{\beta}{\alpha} = \frac{\text{U}\beta}{\text{U}\alpha} = 1 : \frac{\text{U}\alpha}{\text{U}\beta}$$

$$\therefore \ \text{U}\frac{1}{q} = \frac{1}{\text{U}q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (83),$$

and, Equation (82),

$$\text{UK}q = \frac{1}{\text{U}q}.$$

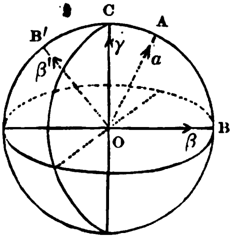
Again, since the conjugate of a versor is the same as the reciprocal of that versor, we have, from Equations (82) and (83),

$$\text{UK}q = \text{KU}q \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

49. *Representation of Versors by spherical arcs.*

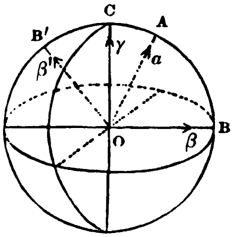
If α, β, γ, ... are co-initial unit vectors, their extremities will all lie on the surface of a unit sphere (Fig. 38). $\frac{\alpha}{\beta}$ being any quaternion, $\text{U}\frac{\alpha}{\beta}$ turns β from the position OB to OA, and this versor may be represented by the arc BA joining the vector extremities; for this arc determines the plane of the versor as also the magnitude and direction of its angle, the direction of rotation being indicated by the order of the letters as in the case of vectors. This representation of versors by vector arcs is of importance in the theorems relating to the multiplication and division of quaternions, and may be made upon a *unit* sphere; for, if α, β, γ, ... are not unit vectors, the quaternions will differ from the versors by a numerical factor only, the introduction of which cannot affect the

Fig. 38.



versor conclusions. Disregarding, then, the tensors, since versors are equal whose planes are parallel and angles equal (including direction), equal arcs on the same great circle and estimated in the same direction represent equal versors, for any arc may be slid over the great circle on which it lies without change of length or reversal of direction. On this plan B'A = AB will represent the reciprocal or conjugate of BA, and a quadrantal versor would have for its representative BC, an arc of 90°. Also, the versors of all complanar quaternions will be represented by arcs of the same great circle, while arcs of different great circles will represent the versors of diplanar quaternions, which are always unequal.

Fig. 38.



If M, N and P are the vertices of a spherical triangle, the vector arcs MN, NP and PM will represent versors, and it will be seen that by taking the geometric sum of two of these arcs in a certain order, the remaining arc will represent the versor of their product; so that if q' be represented by PM and q by NP, $q'q$ may be constructed by a process of spherical addition represented by PM + NP = NM, NM representing the versor $q'q$; but that because $q'q$ and qq' are not generally equal, this process of spherical addition, as representing versor multiplication, is not commutative as was that of vector addition, PM + NP and NP + PM representing diplanar versors.

50. Addition and Subtraction of Quaternions.

Since a quaternion is the sum of a scalar and a vector, in finding the sum or difference of several quaternions the sum or difference of their scalar and vector parts may be taken separately. The former will be a scalar and the latter a vector; consequently, *the sum or difference of several quaternions is a quaternion.*

1. Both the associative and commutative principles being applicable to the summation of scalars, as also to that of vectors (Arts. 4, 5), they also hold good for the addition and subtraction of quaternions; or

$$\left.\begin{array}{r} q+r=r+q \\ \text{and} \quad q+(r+s)=(q+r)+s \end{array}\right\} \quad . \quad . \quad . \quad . \quad (85).$$

If then

$$q = \mathbf{S}q + \mathbf{V}q$$
$$r = \mathbf{S}r + \mathbf{V}r$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$s = q + r + \cdots\cdots = \mathbf{S}s + \mathbf{V}s;$$

in which

$$\mathbf{S}s = \mathbf{S}(q+r+\cdots\cdots) = \mathbf{S}q + \mathbf{S}r + \cdots\cdots,$$
$$\mathbf{V}s = \mathbf{V}(q+r+\cdots\cdots) = \mathbf{V}q + \mathbf{V}r + \cdots\cdots,$$

and, in general,

$$\left.\begin{array}{r} \mathbf{S}\Sigma q = \Sigma\mathbf{S}q \\ \mathbf{V}\Sigma q = \Sigma\mathbf{V}q \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86),$$

or, *in quaternion addition and subtraction*, **S** *and* **V** *are distributive symbols.*

2. If $q + r + p + \cdots\cdots = s$, then, Equation (78),

$$\mathbf{K}q + \mathbf{K}r + \mathbf{K}p + \cdots\cdots = \mathbf{S}q + \mathbf{S}r + \mathbf{S}p + \cdots\cdots - \mathbf{V}q - \mathbf{V}r - \mathbf{V}p - \cdots\cdots$$
$$= \mathbf{S}s - \mathbf{V}s = \mathbf{K}s.$$

$$\therefore \; \Sigma\mathbf{K}q = \mathbf{K}\Sigma q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

K, like **S** and **V**, being a distributive symbol.

3. Again, since the conjugate of a scalar is the scalar itself,

$$\mathbf{KS}q = \mathbf{S}q.$$

But $\mathbf{S}q = \mathbf{SK}q$. Hence

$$\mathbf{KS}q = \mathbf{S}q = \mathbf{SK}q \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

Also, since the conjugate of a vector is the vector reversed,

$$\mathbf{KV}q = -\mathbf{V}q.$$

But $-\mathbf{V}q = \mathbf{VK}q$. Hence

$$\mathbf{KV}q = -\mathbf{V}q = \mathbf{VK}q \quad . \quad . \quad . \quad . \quad . \quad (89);$$

hence **K** *is commutative with* **S** *and* **V**.

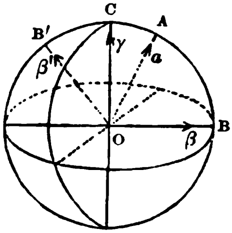


Fig. 38.

4. Since any two quaternions may be reduced to a common denominator (Art. 44), so that

$$\frac{\alpha}{\beta} + \frac{\gamma}{\delta} = \frac{\alpha' + \gamma'}{\delta'},$$

and since

$$\mathbf{T}\alpha' + \mathbf{T}\gamma' > \mathbf{T}(\alpha' + \gamma')$$

unless $\alpha' = x\gamma'$ and $x > 0$, it follows that

$$\mathbf{T}q + \mathbf{T}q' > \mathbf{T}(q + q')$$

unless $q = xq'$ and $x > 0$. Hence, in general, $\mathbf{T}\Sigma q$ is not equal to $\Sigma\mathbf{T}q$. Moreover, since $\mathbf{U}\Sigma q$ is a function of the tensors under the Σ sign, while $\Sigma\mathbf{U}q$ is independent of the tensors, $\mathbf{U}\Sigma q$ is not equal to $\Sigma\mathbf{U}q$. This also appears from the representation of versors by spherical arcs (Fig. 38). Hence, *in the addition and subtraction of quaternions*, **T** *and* **U** *are not, in general, distributive symbols.*

51. Multiplication of Quaternions.

1. Let

$$q = \mathbf{S}q + \mathbf{V}q, \quad r = \mathbf{S}r + \mathbf{V}r$$

be any two quaternions. Then

$$p = qr = \mathbf{S}q\mathbf{S}r + \mathbf{S}q\mathbf{V}r + \mathbf{S}r\mathbf{V}q + \mathbf{V}q\mathbf{V}r.$$

The last member, being the sum of a scalar and a vector, is a quaternion. Hence, *the product of two quaternions is a quaternion*, and

$$p = \mathbf{S}p + \mathbf{V}p = \mathbf{S}qr + \mathbf{V}qr,$$

in which

$$\mathbf{S}qr = \mathbf{S}q\mathbf{S}r + \mathbf{S} \,.\, \mathbf{V}q\mathbf{V}r \quad . \quad . \quad . \quad . \quad (90),$$

and

$$\mathbf{V}qr = \mathbf{S}q\mathbf{V}r + \mathbf{S}r\mathbf{V}q + \mathbf{V} \,.\, \mathbf{V}q\mathbf{V}r \quad . \quad . \quad . \quad (91).$$

If we multiply q by r, we obtain

$$\mathrm{S}rq = \mathrm{S}r\mathrm{S}q + \mathrm{S}\,.\,\mathrm{V}r\mathrm{V}q,$$
$$\mathrm{V}rq = \mathrm{S}r\mathrm{V}q + \mathrm{S}q\mathrm{V}r + \mathrm{V}\,.\,\mathrm{V}r\mathrm{V}q.$$

But, Equation (53),

$$\mathrm{S}\,.\,\mathrm{V}r\mathrm{V}q = \mathrm{S}\,.\,\mathrm{V}q\mathrm{V}r.$$
$$\therefore\ \mathrm{S}rq = \mathrm{S}qr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92).$$

But, Equation (54),

$$\mathrm{V}\,.\,\mathrm{V}q\mathrm{V}r = -\mathrm{V}\,.\,\mathrm{V}r\mathrm{V}q,$$

and therefore the products qr and rq are not equal. Hence, *quaternion multiplication is not in general commutative.* If, however, q and r are complanar, $\mathrm{V}q$ and $\mathrm{V}r$ are parallel, and $\mathrm{V}\,.\,\mathrm{V}q\mathrm{V}r = 0$; in which case $qr = rq$. Conversely, if $qr = rq$, q and r are complanar.

Since Reciprocal, Conjugate and Opposite quaternions are complanar, they are commutative, or

$$\left.\begin{aligned} q\mathrm{K}q &= \mathrm{K}q\,.\,q \\ q\frac{1}{q} &= \frac{1}{q}q = qq^{-1} = q^{-1}q \\ q(-q) &= -qq \end{aligned}\right\} \quad . \quad . \quad . \quad (93).$$

2. It has been shown (Art. 44) that any two quaternions q, q', can be reduced to the forms $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\alpha}$ having a common denominator, or to the forms $\frac{\alpha}{\delta}$ and $\frac{\gamma}{\alpha}$. Hence

$$q' : q = \frac{\gamma}{\alpha} : \frac{\beta}{\alpha} = \frac{\gamma}{\alpha}\,.\,\frac{\alpha}{\beta} = \frac{\gamma}{\beta}.$$

We have then

$$\left.\begin{aligned} \mathrm{T}\frac{q'}{q} = \mathrm{T}\frac{\gamma}{\beta} = \frac{\mathrm{T}\gamma}{\mathrm{T}\beta} = \frac{\mathrm{T}\gamma}{\mathrm{T}\alpha}\,.\,\frac{\mathrm{T}\alpha}{\mathrm{T}\beta} = \frac{\mathrm{T}\gamma}{\mathrm{T}\alpha} : \frac{\mathrm{T}\beta}{\mathrm{T}\alpha} = \mathrm{T}q' : \mathrm{T}q \\ \mathrm{U}\frac{q'}{q} = \mathrm{U}\frac{\gamma}{\beta} = \frac{\mathrm{U}\gamma}{\mathrm{U}\beta} = \frac{\mathrm{U}\gamma}{\mathrm{U}\alpha}\,.\,\frac{\mathrm{U}\alpha}{\mathrm{U}\beta} = \frac{\mathrm{U}\gamma}{\mathrm{U}\alpha} : \frac{\mathrm{U}\beta}{\mathrm{U}\alpha} = \mathrm{U}q' : \mathrm{U}q \end{aligned}\right\} \quad (94).$$

In a similar manner

$$\left.\begin{aligned}\mathrm{T}(q'q)&=\mathrm{T}\left[\frac{\gamma}{\alpha}\cdot\frac{\alpha}{\delta}\right]=\mathrm{T}\frac{\gamma}{\delta}=\frac{\mathrm{T}\gamma}{\mathrm{T}\delta}=\frac{\mathrm{T}\gamma}{\mathrm{T}\alpha}\cdot\frac{\mathrm{T}\alpha}{\mathrm{T}\delta}=\mathrm{T}q'\cdot\mathrm{T}q\\ \mathrm{U}(q'q)&=\mathrm{U}\frac{\gamma}{\delta}=\mathrm{U}q'\mathrm{U}q\end{aligned}\right\}\quad(95).$$

Hence *the tensor of the product (or quotient) of any two quaternions is the product (or quotient) of their tensors, and the versor of the product (or quotient) is the product (or quotient) of their versors.*

In fact, tensors being commutative, we have, in general,

$$\mathrm{T}\Pi q=\Pi\mathrm{T}q \quad . \; . \; . \; . \; . \; . \quad (96),$$

$$\Pi q=\mathrm{T}\Pi q\cdot\mathrm{U}\Pi q=\Pi\mathrm{T}q\cdot\Pi\mathrm{U}q,$$

$$\therefore\ \mathrm{U}\Pi q=\Pi\mathrm{U}q \quad . \; . \; . \; . \; . \; . \quad (97).$$

3. The multiplication and division of tensors being purely arithmetical operations, we proceed to the corresponding operations on the versors. It has been shown (Art. 44) that any two versors q, q' may be reduced to the forms

$$q=\frac{\beta}{\alpha}=\frac{\mathrm{OB}}{\mathrm{OA}},\quad q'=\frac{\gamma'}{\beta}=\frac{\mathrm{OC}'}{\mathrm{OB}},\qquad \text{(Fig. 39)},$$

A, B, C$'$ being the vertices of a spherical triangle on a unit sphere. Then

$$q'q=\frac{\gamma'}{\beta}\cdot\frac{\beta}{\alpha}=\frac{\gamma'}{\alpha}=\frac{\mathrm{OC}'}{\mathrm{OA}}.$$

If we represent the versors q' and q by the vector arcs BC$'$ and AB, then the versor $\frac{\gamma'}{\alpha}$, the product of $q'q$, will be represented by the arc AC$'$; moreover if $q''=\frac{\gamma'}{\alpha}$ represent any dividend and $q=\frac{\beta}{\alpha}$ any divisor, then

$$\frac{q''}{q}=\frac{\gamma'}{\alpha}\cdot\frac{\alpha}{\beta}=\frac{\gamma'}{\beta}=\frac{\mathrm{OC}'}{\mathrm{OB}};$$

the versor of the product $q'q$ being

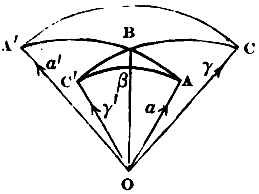
$$\mathrm{BC}'+\mathrm{AB}=\mathrm{AC}',$$

and the versor of the quotient $\frac{q''}{q}$

$$AC' - AB = BC';$$

and, as in the addition and subtraction of quaternions, the process consisted in an algebraic addition and subtraction of scalars but a geometric addition and subtraction of vectors, so the multiplication and division of quaternions is reduced to the corresponding arithmetical operations on the tensors and the geometrical multiplication and division of the versors, the latter being constructed by means of representative arcs and the rules of spherical addition and subtraction.

Fig. 39.



4. The representation of a versor by the arc of a great circle on a unit sphere illustrates the non-commutative character of quaternion multiplication. For, AB and BA' (Fig. 39) being equal arcs on the same great circle, as versors

$$AB = BA',$$

and similarly

$$CB = BC'.$$

Now if

$$q = \frac{\beta}{a} = \frac{a'}{\beta} \quad \text{and} \quad r = \frac{\beta}{\gamma} = \frac{\gamma'}{\beta},$$

then

$$qr = \frac{a'}{\beta}\frac{\beta}{\gamma} = \frac{a'}{\gamma} \quad \text{and} \quad rq = \frac{\gamma'}{\beta}\frac{\beta}{a} = \frac{\gamma'}{a},$$

the versors qr and rq being represented by the arcs CA' and AC' respectively. These arcs, though equal in length, are not in the same plane, and therefore the versors rq and qr are not equal. Constructing these versors, by spherical addition we should have

$$BC' + AB = AC',$$

$$AB + BC' = BA' + CB = CA',$$

a change in the order giving unequal results.

Hence, unless AC′ and CA′ lie on the same great circle, in which case q and r are complanar, quaternion multiplication is not commutative.

5. Other results, hereafter to be obtained symbolically, may be readily proved by means of spherical arcs, as follows:

If AB (Fig. 39) represents the versor of $q = \frac{\beta}{\alpha}$, A′B = BA represents the versor of $\mathbf{K}q$ or $\frac{1}{q}$. The spherical sum of AB + BA being zero, the effect of the versors in the products $q\mathbf{K}q$ and $q\frac{1}{q}$ is to annul each other. Hence, if the vectors are not unit vectors,

$$q\mathbf{K}q = \mathbf{K}q \cdot q = (\mathbf{T}q)^2 \quad . \quad . \quad . \quad . \quad . \quad (98),$$

$$q\frac{1}{q} = \frac{1}{q}q = 1.$$

Again, from

$$\text{AB} + \text{BC}' = \text{CA}',$$

we have

$$qr = \frac{\alpha'}{\gamma},$$

and the versor of $\mathbf{K}\,(qr)$ will therefore be represented by A′C.

But

$$\text{A}'\text{C} = \text{BC} + \text{A}'\text{B},$$

whence

$$\mathbf{K}(qr) = \mathbf{K}r\mathbf{K}q \quad . \quad . \quad . \quad . \quad . \quad . \quad (99),$$

or, *the conjugate of the product of two quaternions is the product of their conjugates in inverted order.*

6. The product or quotient of complanar quaternions is readily derived from the foregoing explanation of versor products and quotients as dependent upon a geometric composition of rotations. For, disregarding the tensors, the vector arcs which represent the versors, since the latter are complanar, will lie on the same great circle, and the processes which for diplanar versors were geometric now become algebraic. Thus for $q' = \frac{\alpha'}{\beta}$ and $q = \frac{\beta}{\alpha}$,

$$qq' = q'q = \frac{\alpha'}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\alpha'}{\alpha},$$

and, Fig. 39,

$$\text{BA}' + \text{AB} = \text{AB} + \text{BA}' = \text{AA}';$$

also for $q'' = \frac{\alpha'}{\alpha}$ and $q' = \frac{\beta}{\alpha}$,

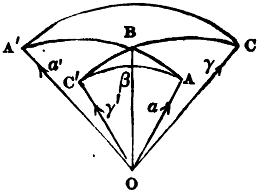
$$\frac{q''}{q'} = \frac{\alpha'}{\alpha} : \frac{\beta}{\alpha} = \frac{\alpha'}{\alpha} \cdot \frac{\alpha}{\beta} = \frac{\alpha'}{\beta},$$

and

$$\text{BA} + \text{AA}' = \text{BA}'.$$

The product or quotient of any two complanar quaternions is therefore obtained by *multiplying or dividing their tensors and adding or subtracting their angles.* Thus

Fig. 39.



$$pq = \text{T}p \cdot \text{T}q\,[\cos(\phi + \theta) + \epsilon \sin(\phi + \theta)].$$

If $p = q$,

$$q^2 = (\text{T}q)^2 (\cos 2\phi + \epsilon \sin 2\phi),$$

or, generally,

$$q^n = (\text{T}q)^n (\cos n\phi + \epsilon \sin n\phi) \quad . \quad . \quad . \quad (100),$$

whence result the following general formulae,

$$\left.\begin{aligned} \text{T}(q^n) &= (\text{T}q)^n \\ \text{U}(q^n) &= (\text{U}q)^n \\ \text{SU}(q^n) &= \cos n\angle q \\ \text{TVU}(q^n) &= \sin n\angle q \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (101),$$

which are all involved in Art. 42.

52. 1. **Distributive** *and* **Associative Laws** *in Vector and Quaternion Multiplication.*

Having assumed (Art. 24)

$$\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = \frac{\beta + \gamma}{\alpha},$$

whence

$$\beta\alpha^{-1} + \gamma\alpha^{-1} = (\beta + \gamma)\,\alpha^{-1},$$

since α is any vector, we have

$$\beta\alpha + \gamma\alpha = (\beta + \gamma)\,\alpha. \qquad (a)$$

Taking the conjugate of $(\beta + \gamma)\alpha$,

$$\begin{aligned} \mathbf{K}[(\beta + \gamma)\alpha] &= \mathbf{K}\alpha\mathbf{K}(\beta + \gamma) && \text{[Eq. 99]} \\ &= \mathbf{K}\alpha(\mathbf{K}\beta + \mathbf{K}\gamma). && \text{[Eq. 87]} \end{aligned}$$

Taking the conjugate of $(\beta\alpha + \gamma\alpha)$,

$$\mathbf{K}(\beta\alpha + \gamma\alpha) = \mathbf{K}\beta\alpha + \mathbf{K}\gamma\alpha = \mathbf{K}\alpha\mathbf{K}\beta + \mathbf{K}\alpha\mathbf{K}\gamma.$$

Hence

$$\mathbf{K}\alpha(\mathbf{K}\beta + \mathbf{K}\gamma) = \mathbf{K}\alpha\mathbf{K}\beta + \mathbf{K}\alpha\mathbf{K}\gamma,$$

or

$$\alpha'(\beta' + \gamma') = \alpha'\beta' + \alpha'\gamma'. \qquad (b)$$

Hence, from (a) and (b), *the multiplication of vectors is a doubly distributive operation*, and

$$(\beta + \gamma)(\alpha + \delta) = \beta\alpha + \gamma\alpha + \beta\delta + \gamma\delta \quad . \quad . \quad (102).$$

2. Let $q = \frac{\beta'}{\delta'}$ be any quaternion and α any vector; also β a vector along the line of intersection of a plane perpendicular to α with the plane of q. Then another vector, δ, may be found in the latter plane, such that $q = \frac{\beta}{\delta}$, $\frac{\beta}{\delta}$ having the same angle, plane and axis as $\frac{\beta'}{\delta'}$. Also let γ be a vector in the intersecting plane, such that $\frac{\gamma}{\beta} = \alpha$. If now a be any scalar,

$$\begin{aligned} (a + \alpha)q &= \left(a + \frac{\gamma}{\beta}\right)\frac{\beta}{\delta} = \left(\frac{a\beta}{\beta} + \frac{\gamma}{\beta}\right)\frac{\beta}{\delta} \\ &= \frac{a\beta + \gamma}{\beta} \cdot \frac{\beta}{\delta} = \frac{a\beta + \gamma}{\delta} \\ &= a\frac{\beta}{\delta} + \frac{\gamma}{\delta} = a\frac{\beta}{\delta} + \frac{\gamma}{\beta} \cdot \frac{\beta}{\delta} \\ &= aq + \alpha q. \end{aligned}$$

Taking the conjugates as above,

$$q'(a' + \alpha') = q'a' + q'\alpha'.$$

Hence, in general,

$$(a + \alpha)(a' + \alpha') = aa' + a\alpha' + a'\alpha + \alpha\alpha'; \qquad (c)$$

or regarding a, a', and α, α' each as the sum of two scalars and two vectors respectively,

$$(a_1 + a_2 + \alpha_1 + \alpha_2)(a'_1 + a'_2 + \alpha'_1 + \alpha'_2) =$$

$$(a_1 + a_2)(a'_1 + a'_2) + (a_1 + a_2)(\alpha'_1 + \alpha'_2) + (a'_1 + a'_2)$$

$$(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2)(\alpha'_1 + \alpha'_2) =$$

$$(a_1 + \alpha_1)(a'_1 + \alpha'_1) + (a_1 + \alpha_1)(a'_2 + \alpha'_2) + (a_2 + \alpha_2)$$

$$(a'_1 + \alpha'_1) + (a_2 + \alpha_2)(a'_2 + \alpha'_2),$$

since, from (c), the factors in the expression preceding the last are distributive. Putting for the parentheses, which are sums of a scalar and a vector, the quaternion symbols p, q, r and s, we have

$$(p + q)(r + s) = pr + ps + qr + qs \quad . \quad . \quad (103),$$

or, *the multiplication of quaternions is a doubly distributive operation.*

3. Assuming any three quaternions under the quadrinomial form, Article 43, i, k, j being unit vectors along three mutually rectangular axes, we have

$$q = w + xi + yj + zk, \qquad (a)$$

$$r = w' + x'i + y'j + z'k, \qquad (b)$$

$$s = w'' + x''i + y''j + z''k. \qquad (c)$$

Multiplying first (c) by (b) and the result by (a), and then (b) by (a) and (c) by this result, observing the order of the factors, it will be found that the scalar and vector parts of these two products are respectively equal, and therefore

$$q(rs) = (qr)s \quad . \quad . \quad . \quad . \quad . \quad (104),$$

or, *the associative law is true in the multiplication of quaternions.*

53. 1. If α and β be any two vectors, then

$$(\alpha+\beta)(\alpha+\beta)=(\alpha+\beta)^2=\alpha^2+(\alpha\beta+\beta\alpha)+\beta^2,$$

whence, Equation (55), or, comparing Equations (39), (41) and (80),

$$(\alpha+\beta)^2=\alpha^2+2\,\mathrm{S}\alpha\beta+\beta^2 \quad . \quad . \quad . \quad (105).$$

2. Similarly

$$(\alpha-\beta)(\alpha-\beta)=(\alpha-\beta)^2=\alpha^2-(\alpha\beta+\beta\alpha)+\beta^2,$$

or

$$(\alpha-\beta)^2=\alpha^2-2\,\mathrm{S}\alpha\beta+\beta^2 \quad . \quad . \quad . \quad (106).$$

3. From Equation (57), or by multiplying $q=\mathrm{S}q+\mathrm{V}q$ by $\mathrm{K}q=\mathrm{S}q-\mathrm{V}q$,

$$\alpha\beta\,.\,\beta\alpha=(\mathrm{S}q)^2-(\mathrm{V}q)^2;$$

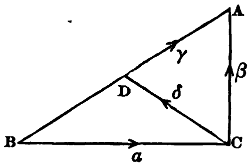
hence, from Equation (98), the equalities

$$\alpha\beta\,.\,\beta\alpha=q\mathrm{K}q=(\mathrm{S}\alpha\beta)^2-(\mathrm{V}\alpha\beta)^2=(\mathrm{T}q)^2 \quad . \quad (107).$$

54. Applications.

1. *In any right-angled triangle, the square on the hypothenuse is equal to the sum of the squares on the sides.*

Fig. 40.



Let the sides, as vectors, be represented by α and β (Fig. 40), and the hypothenuse by γ. Then

$$\gamma=\alpha+\beta.$$

Squaring, Equation (105),

$$\gamma^2=\alpha^2+2\,\mathrm{S}\alpha\beta+\beta^2,$$

or, Art. 41, 4,

$$\gamma^2=\alpha^2+\beta^2,$$

or, as lengths simply, changing signs [Equation (33)],

$$\mathrm{BA}^2=\mathrm{BC}^2+\mathrm{CA}^2.$$

2. *In any right-angled triangle, the medial to the hypothenuse is one-half the hypothenuse.*

In Fig. 40, for the medial vector CD $= \delta$, we have (Art. 15)

$$\delta = \tfrac{1}{2}(\beta - \alpha),$$

or

$$2\delta = \beta - \alpha.$$

Squaring, and since $S\beta\alpha = 0$,

$$4\delta^2 = \beta^2 + \alpha^2,$$

or

$$\text{CD}^2 = \frac{\text{CA}^2 + \text{CB}^2}{4} = \frac{\text{AB}^2}{4};$$

$$\therefore \text{CD} = \frac{\text{AB}}{2}.$$

3. *If the diagonals of a parallelogram are at right angles to each other, it is a rhombus.*

Let the vector sides be represented by α and β. Then $\alpha + \beta$ and $\alpha - \beta$ are the vector diagonals.

By condition

$$S(\alpha + \beta)(\alpha - \beta) = 0. \qquad \text{[Art. 41, 4]}$$

But, Equation (53),

$$S(\alpha + \beta)(\alpha - \beta) = \alpha^2 - \beta^2 = 0,$$

which is true only when $T\alpha = T\beta$, that is when the sides are equal.

4. *The figure formed by joining the middle points of the sides of a square is itself a square.*

Let BC and CA (Fig. 40) be the sides of a square, P and Q their middle points, and O the middle point of the side opposite BC. Then, with the same notation,

$$\text{PQ} = \tfrac{1}{2}(\alpha + \beta), \qquad \text{QO} = \tfrac{1}{2}(\beta - \alpha);$$

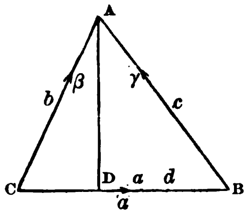
$$\therefore S(\text{PQ} \cdot \text{QO}) = 0,$$

or PQ and QO are at right angles.

5. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other sides, less*

twice the product of the base and the line between the acute angle and the foot of a perpendicular from the angle opposite the base.

Fig. 41.



Let CA $= \beta$, CB $= \alpha$, BA $= \gamma$ (Fig. 41). Then

$$\beta = \alpha + \gamma,$$
$$\beta^2 = \alpha^2 + 2\,\mathrm{S}\alpha\gamma + \gamma^2;$$

Now

$$2\,\mathrm{S}\alpha\gamma = -2\,\mathrm{T}\alpha\mathrm{T}\gamma\cos(180 - \mathrm{B}) = 2ac\cos\mathrm{B}.$$

Hence

$$-b^2 = -a^2 - c^2 + 2ac\cos\mathrm{B} = -a^2 - c^2 + 2ad,$$

or

$$b^2 = a^2 + c^2 - 2ad.$$

If B is a right angle, $\mathrm{S}\alpha\gamma = 0$, and, as in Example 1,

$$b^2 = a^2 + c^2.$$

What does this theorem become for a side opposite an *obtuse* angle?

6. *In any plane triangle, to find a side in terms of the other two sides and their opposite angles.*

In Fig. 41,

$$\beta = \alpha + \gamma.$$

Multiplying into α

$$\beta\alpha = \alpha^2 + \gamma\alpha.$$

Taking the scalars (Art. 41, 5),

$$\mathrm{S}\beta\alpha = -a^2 + \mathrm{S}\gamma\alpha,$$

or

$$-ba\cos\mathrm{C} = -a^2 - ca\cos(180^\circ - \mathrm{B})\,;$$
$$\therefore\ a = b\cos\mathrm{C} + c\cos\mathrm{B}.$$

The above operation with α is indicated by saying simply, "operating with $\times\,\mathrm{S}\,.\,\alpha$," meaning that α is first introduced and then the scalars taken. The position of the sign × will indicate *how* α is used. If used as a multiplier, we should write, "operating with $\mathbf{S}\,.\,\alpha\times$."

7. *The sines of the angles, in any plane triangle, are proportional to the opposite sides.*

In Fig. 41

$$\beta=\alpha+\gamma.$$

Operating with $\times\,\mathbf{V}\,.\,\alpha$, that is, as explained in the preceding example, multiplying into α and taking the vectors (Art. 41, 5),

$$\mathbf{V}\beta\alpha=\mathbf{V}(\alpha+\gamma)\,\alpha=\mathbf{V}\,.\,\alpha^2+\mathbf{V}\gamma\alpha.$$

But $\mathbf{V}\,.\,\alpha^2=0$; hence

$$\mathbf{V}\beta\alpha=\mathbf{V}\gamma\alpha,$$

$$ba\,\sin\mathrm{C}=ca\,\sin\mathrm{B},$$

or

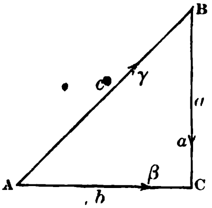
$$\sin\mathrm{C}:\sin\mathrm{B}::c:b.$$

Notice that $\mathbf{V}\beta\alpha$ and $\mathbf{V}\gamma\alpha$ involve a unit vector at right angles to their plane, and that, owing to the *order* of the vector factors, ϵ has the same sign in both members of the equality, and may therefore be cancelled. The period in $\mathbf{V}\,.\,\alpha^2$ may evidently be omitted, as in $\mathbf{V}\beta\alpha$; it will be used hereafter only to avoid ambiguity. Thus $\mathbf{K}qr$ means the conjugate of qr; but $\mathbf{K}q\,.\,r$ is r multiplied by the conjugate of q.

8. *In a right-angled triangle, to find the sine and cosine of the acute angles.*

Let $\mathrm{AB}=\gamma$, $\mathrm{AC}=\beta$, $\mathrm{BC}=\alpha$ (Fig. 42).

Fig. 42.



Then

$$\beta=\gamma+\alpha,$$

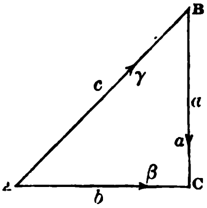
whence

$$1=\frac{\gamma}{\beta}+\frac{\alpha}{\beta}.$$

Taking the scalars, since $\mathbf{S}\dfrac{\alpha}{\beta}=0$,

$$1=\frac{c}{b}\cos\mathrm{A},\quad\text{or}\quad\cos\mathrm{A}=\frac{b}{c}.$$

Fig. 42.



Taking the vectors

$$V\frac{\gamma}{\beta}+V\frac{\alpha}{\beta}=0,$$

$$\frac{c}{b}\sin A-\frac{a}{b}\sin C=0;$$

$$\therefore\ \sin A=\frac{a}{c}.$$

In this example $UV\frac{\gamma}{\beta}=-UV\frac{\alpha}{\beta}$.

9. *To find the sine and cosine of the sum of two angles.*

Let α, β, γ be complanar unit vectors (Fig. 43), and ϵ a unit vector perpendicular to their plane. We have

$$\frac{\gamma}{\alpha}=\frac{\gamma}{\beta}\cdot\frac{\beta}{\alpha},$$

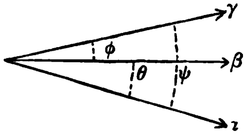
in which

$$\frac{\gamma}{\alpha}=\cos(\phi+\theta)+\epsilon\sin(\phi+\theta),$$

$$\frac{\gamma}{\beta}=\cos\phi+\epsilon\sin\phi,$$

$$\frac{\beta}{\alpha}=\cos\theta+\epsilon\sin\theta.$$

Fig. 43.



Hence

$$\begin{aligned}\cos(\phi+\theta)+\epsilon\sin(\phi+\theta)&=(\cos\phi+\epsilon\sin\phi)(\cos\theta+\epsilon\sin\theta)\\&=\cos\phi\cos\theta+\epsilon(\sin\phi\cos\theta+\cos\phi\sin\theta)+\epsilon^2\sin\theta\sin\phi.\end{aligned}$$

Equating the scalar and vector parts in succession, there results, since $\epsilon^2=-1$,

$$\cos(\phi+\theta)=\cos\phi\cos\theta-\sin\phi\sin\theta,$$
$$\sin(\phi+\theta)=\sin\phi\cos\theta+\cos\phi\sin\theta.$$

10. *To find the sine and cosine of the difference of two angles.*

Let the angle between γ and α (Fig. 43) be ψ. Then

$$\frac{\beta}{\gamma}=\frac{\beta}{\alpha}\cdot\frac{\alpha}{\gamma},$$

in which

$$\frac{\beta}{\gamma} = \cos(\psi - \theta) - \epsilon \sin(\psi - \theta),$$
$$\frac{\beta}{\alpha} = \cos\theta + \epsilon \sin\theta,$$
$$\frac{\alpha}{\gamma} = \cos\psi - \epsilon \sin\psi,$$

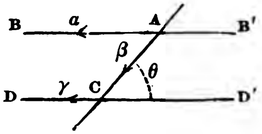
and, as in the preceding example,

$$\cos(\psi - \theta) = \cos\theta \cos\psi + \sin\theta \sin\psi,$$
$$\sin(\psi - \theta) = \cos\theta \sin\psi - \sin\theta \cos\psi.$$

11. *If a straight line intersect two other straight lines so as to make the alternate angles equal, the two lines are parallel.*

Let α and γ (Fig. 44) be unit vectors along AB and CD, and β a unit vector along AC. Then

Fig. 44.



$$\alpha\beta = -\cos\theta + \epsilon\sin\theta,$$
$$\beta\gamma = -\cos\theta - \epsilon\sin\theta;$$

whence

$$\alpha\beta - \beta\gamma = 2\,\mathbf{V}\alpha\beta,$$

and therefore, Equation (56), $\gamma = \alpha$.

If $\alpha =$ AB′, then

$$\alpha\beta = \cos\theta - \epsilon\sin\theta,$$
$$\beta\gamma = -\cos\theta - \epsilon\sin\theta,$$
$$\alpha\beta - \beta\gamma = 2\,\mathbf{S}\alpha\beta;$$
$$\therefore\ \gamma = -\alpha. \qquad \text{[Eq. (55)]}$$

12. *If a parallelogram be described on the diagonals of any parallelogram, the area of the former is twice that of the latter.*

Let α and β represent the sides as vectors; then the diagonals are $\alpha + \beta$ and $\alpha - \beta$, and

$$\mathbf{V}(\alpha + \beta)(\alpha - \beta) = \mathbf{V}(\beta\alpha - \alpha\beta) = 2\,\mathbf{V}\beta\alpha,$$

since $\mathbf{V}\alpha^2 = \mathbf{V}\beta^2 = 0$ and $-\mathbf{V}\alpha\beta = \mathbf{V}\beta\alpha$.

But, from the order of the factors,

$$\mathbf{UV}(\alpha+\beta)(\alpha-\beta)=\mathbf{UV}\beta\alpha,$$

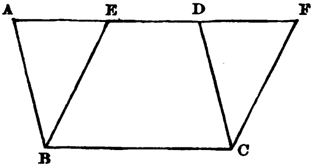
hence

$$\mathbf{TV}(\alpha+\beta)(\alpha-\beta)=2\,\mathbf{TV}\beta\alpha,$$

which is the proposition (Art. 41, 7).

13. *Parallelograms on the same base and between the same parallels are equal.*

Fig. 45.



We have (Fig. 45)

$$\mathrm{BE}=\mathrm{BA}+\mathrm{AE}$$
$$=\mathrm{BA}+x\,\mathrm{BC}.$$

Operating with $\mathbf{V}\,.\,\mathrm{BC}\times$

$$\mathbf{V}(\mathrm{BC}\,.\,\mathrm{BE})=\mathbf{V}(\mathrm{BC}\,.\,\mathrm{BA}),$$

since $\mathbf{V}x\mathrm{BC}^2=0$.

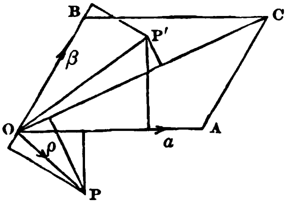
Hence

$$\mathrm{BC}\,.\,\mathrm{BE}\sin\mathrm{EBC}=\mathrm{BC}\,.\,\mathrm{BA}\sin\mathrm{ABC},$$

which is also true when the bases are equal, but not co-incident.

14. *If, from any point in the plane of a parallelogram, perpendiculars are let fall on the diagonal and the two sides that contain it, the product of the diagonal and its perpendicular is equal to the sum, or difference, of the products of the sides and their respective perpendiculars, as the point lies without or within the parallelogram.*

Fig. 46.



Let $\mathrm{OA}=\alpha$, $\mathrm{OB}=\beta$, $\mathrm{OP}=\rho$ (Fig. 46).

Then

$$\mathbf{V}\alpha\rho+\mathbf{V}\beta\rho=\mathbf{V}(\alpha+\beta)\rho.$$

But

$$\mathbf{UV}\alpha\rho=\mathbf{UV}\beta\rho=\mathbf{UV}(\alpha+\beta)\rho.$$

Hence

$$\mathbf{TV}\alpha\rho+\mathbf{TV}\beta\rho=\mathbf{TV}(\alpha+\beta)\rho.$$

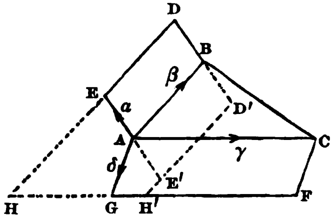
For $\rho' = \mathrm{OP}'$, we have

$$\mathrm{UV}\alpha\rho' = -\mathrm{UV}\beta\rho' = \pm\mathrm{UV}(\alpha+\beta)\rho';$$
$$\therefore\ \mathrm{TV}\alpha\rho' - \mathrm{TV}\beta\rho' = \mathrm{TV}(\alpha+\beta)\rho'.$$

15. *If, on any two sides of a triangle, as* AC, AB (Fig. 47), *any two exterior parallelograms, as* ACFG, ADEB, *be constructed, and the sides* ED, GF, *produced to meet in* H, *then will the sum of the areas of the parallclograms be equal to that whose sides are equal and parallel to* CB *and* AH.

Let $\mathrm{AE} = \alpha$, $\mathrm{AB} = \beta$, $\mathrm{AC} = \gamma$ and $\mathrm{AG} = \delta$. Then

Fig. 47.



$$\mathrm{AH} = \mathrm{AE} + \mathrm{EH}$$
$$= \alpha - x\beta.$$

Operating with $\times\mathrm{V}\,.\,\beta$

$$\mathrm{V}(\mathrm{AH}\,.\,\beta) = \mathrm{V}\alpha\beta. \qquad (a)$$

We have also

$$\mathrm{AH} = \mathrm{AG} + \mathrm{GH}$$
$$= \delta - y\gamma.$$

Operating with $\times\mathrm{V}\,.\,\gamma$

$$\mathrm{V}(\mathrm{AH}\,.\,\gamma) = \mathrm{V}\delta\gamma. \qquad (b)$$

Hence, from (*a*) and (*b*),

$$\mathrm{V}\mathrm{AH}(\beta-\gamma) = \mathrm{V}\alpha\beta - \mathrm{V}\delta\gamma,$$
$$\mathrm{V}(\mathrm{AH}\,.\,\mathrm{CB}) = \mathrm{V}\alpha\beta - \mathrm{V}\delta\gamma = \mathrm{V}\alpha\beta + \mathrm{V}\gamma\delta.$$

These vectors have a common versor; whence the proposition.

If one of the parallelograms, as AD', be interior, then $\mathrm{AE}' = -\alpha$ and $\mathrm{AH}' = -\alpha - x\beta = \delta + y\gamma$, and

$$\mathrm{V}(\mathrm{AH}'\,.\,\beta) = -\mathrm{V}\alpha\beta,$$
$$\mathrm{V}(\mathrm{AH}'\,.\,\gamma) = \mathrm{V}\delta\gamma;$$
$$\therefore\ \mathrm{V}\mathrm{AH}'(\beta-\gamma) = -\mathrm{V}\alpha\beta - \mathrm{V}\delta\gamma = \mathrm{V}\beta\alpha - \mathrm{V}\delta\gamma.$$

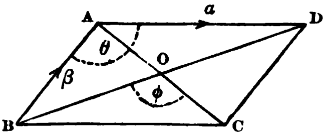
But in this case

$$\mathrm{UV}(\mathrm{AH}' \,.\, \mathrm{CB}) = -\mathrm{UV}\beta\alpha = -\mathrm{UV}\delta\gamma,$$

and the area of the parallelogram on AH′, CB, is the area of AF minus the area of AD′.

16. *To find the angle between the diagonals of a parallelogram.*

Fig. 48.



Let $\mathrm{AD} = \mathrm{BC} = \alpha$ (Fig. 48), and $\mathrm{BA} = \mathrm{CD} = \beta$, d and d' being the tensors of the diagonals. Then

$$\begin{aligned}\mathrm{AC} \,.\, \mathrm{DB} &= -(\alpha - \beta)(\alpha + \beta)\\ &= -\alpha^2 - (\alpha\beta - \beta\alpha) + \beta^2\\ &= -\alpha^2 - 2\,\mathrm{V}\alpha\beta + \beta^2.\end{aligned}$$

Taking the scalars

$$\cos \mathrm{DOC} \,.\, dd' = a^2 - b^2.$$

Taking the vectors

$$\sin \mathrm{DOC} \,.\, dd' = 2ab \sin\theta,$$

since $\mathrm{UV}(\mathrm{AC} \,.\, \mathrm{DB}) = -\mathrm{UV}\alpha\beta$.

$$\therefore\ \tan \mathrm{DOC} = -\tan\phi = \frac{2ab\sin\theta}{a^2 - b^2}.$$

17. *The sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.*

In Fig. 48

$$\begin{aligned}\mathrm{BD}^2 &= (\alpha + \beta)^2 = \alpha^2 + 2\,\mathrm{S}\alpha\beta + \beta^2,\\ \mathrm{CA}^2 &= (\beta - \alpha)^2 = \beta^2 - 2\,\mathrm{S}\alpha\beta + \alpha^2;\\ \therefore\ \mathrm{CA}^2 &+ \mathrm{BD}^2 = 2\alpha^2 + 2\beta^2,\end{aligned}$$

or

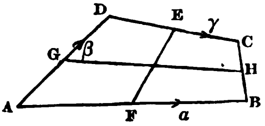
$$\mathrm{BD}^2 + \mathrm{CA}^2 = \mathrm{BA}^2 + \mathrm{AD}^2 + \mathrm{DC}^2 + \mathrm{CB}^2.$$

18. *The sum of the squares of the diagonals of any quadrilateral is twice the sum of the squares of the lines joining the middle points of the opposite sides.*

Let $\text{AB} = \alpha$, $\text{AD} = \beta$, $\text{DC} = \gamma$ (Fig. 49). For the squares of the diagonals, we have

$$(\beta+\gamma)^2+(\beta-\alpha)^2,$$

Fig. 49.



and for the bisecting lines

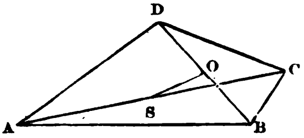
$$[\tfrac{1}{2}\beta+\gamma-\tfrac{1}{2}(\beta+\gamma-\alpha)]^2+[\beta+\tfrac{1}{2}\gamma-\tfrac{1}{2}\alpha]^2.$$

Whence the proposition readily follows.

19. *The sum of the squares of the sides of any quadrilateral exceeds the sum of the squares on the diagonals by four times the square of the line joining the middle points of the diagonals.*

Let $\text{AB} = \alpha$, $\text{AC} = \beta$, $\text{AD} = \gamma$ (Fig. 50). The squares of the sides as vectors are

Fig. 50.



$$\alpha^2+(\beta-\alpha)^2+(\gamma-\beta)^2+\gamma^2,$$

or

$$2(\alpha^2+\beta^2+\gamma^2)-2\,\mathrm{S}\beta\alpha-2\,\mathrm{S}\gamma\beta.$$

The squares of the diagonals are

$$\beta^2+(\gamma-\alpha)^2,$$

or

$$\beta^2+\gamma^2+\alpha^2-2\,\mathrm{S}\gamma\alpha.$$

The former sum exceeds the latter by

$$\alpha^2+\beta^2+\gamma^2-2\,\mathrm{S}\beta\alpha-2\,\mathrm{S}\gamma\beta+2\,\mathrm{S}\gamma\alpha,$$

or by

$$(\alpha-\beta+\gamma)^2,$$

which may be put under the form

$$4\left(\frac{\alpha+\gamma}{2}-\frac{\beta}{2}\right)^2.$$

But $\frac{\alpha+\gamma}{2} = \text{AO}$, and $-\frac{\beta}{2} = \text{SA}$. Substituting these values, we obtain

$$4(\text{AO} + \text{SA})^2, \text{ or } 4\,\text{SO}^2,$$

which is also true of the vector lengths.

20. *In any quadrilateral, if the lines joining the middle points of opposite sides are at right angles, the diagonals are equal.*

With the notation of Fig. 49, we have

$$\text{FE} \cdot \text{GH} = [\tfrac{1}{2}(\gamma - \alpha) + \beta]\tfrac{1}{2}(\alpha + \gamma).$$

But, by condition,

$$S(\text{FE} \cdot \text{GH}) = \frac{\gamma^2}{4} - \frac{\alpha^2}{4} + \frac{S\beta\gamma}{2} + \frac{S\beta\alpha}{2} = 0.$$

Whence

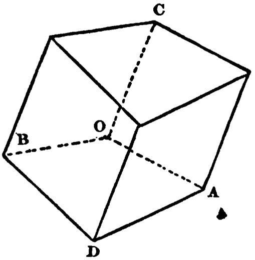
$$(\gamma + \beta)^2 = (\beta - \alpha)^2,$$

$$\text{AC}^2 = \text{BD}^2,$$

or

$$\text{AC} = \text{BD}.$$

Fig. 51.



21. *In any quadrilateral prism, the sum of the squares of the edges exceeds the sum of the squares of the diagonals by eight times the square of the line joining the points of intersection of the two pairs of diagonals.*

Let $\text{OA} = \alpha$, $\text{OB} = \beta$, $\text{OC} = \gamma$, $\text{OD} = \delta$ (Fig. 51). For the sum of the squares of the edges we have

$$2[\alpha^2 + \beta^2 + (\delta - \alpha)^2 + 2\gamma^2 + (\delta - \beta)^2],$$

or

$$2[2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\delta^2 - 2S\delta\alpha - 2S\delta\beta]. \qquad (a)$$

The sum of the squares of the diagonals is

$$(\gamma + \delta)^2 + (\gamma - \delta)^2 + (\gamma + \alpha - \beta)^2 + (\gamma + \beta - \alpha)^2,$$

or

$$2(\alpha^2 + \beta^2 + \delta^2 + 2\gamma^2 - 2S\alpha\beta). \qquad (b)$$

The vectors to the intersections of the diagonals are

$$\tfrac{1}{2}(\delta+\gamma) \quad\text{and}\quad \tfrac{1}{2}(\gamma+\alpha+\beta),$$

and the vector joining these points is

$$\tfrac{1}{2}(\alpha+\beta-\delta).$$

Squaring and multiplying by eight, we have

$$2[\alpha^2+\beta^2+\delta^2+2\,S\alpha\beta-2\,S\alpha\delta-2\,S\beta\delta],$$

which added to (*b*) gives (*a*).

22. *In any tetraedron, if two pairs of opposite edges are at right angles, respectively, the third pair will be at right angles.*

Let OA $=\alpha$, OB $=\beta$, OC $=\gamma$ (Fig. 52).

The conditions give

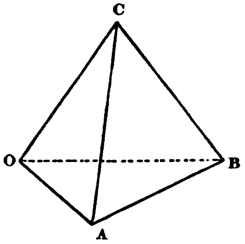
$$S\alpha(\beta-\gamma)=0,$$
$$S\beta(\alpha-\gamma)=0.$$

Subtracting the first of these equations from the second

$$S\gamma(\alpha-\beta)=0,$$

which is the proposition.

Fig. 52.



23. *To find the relations between the edges, angles and plane areas of a tetraedron.*

With the notation of Fig. 52, we have

$$\text{CA}\cdot\text{CB}=(\alpha-\gamma)(\beta-\gamma),$$

or

$$\text{CA}\cdot\text{CB}=\alpha\beta-\alpha\gamma-\gamma\beta+\gamma^2. \qquad (a)$$

Representing the tensors of CA and CB by *m* and *n*, and taking the scalars of (*a*),

$$S(\text{CA}\cdot\text{CB})=S\alpha\beta-S\alpha\gamma-S\gamma\beta+\gamma^2,$$

whence

$$c^2-ac\cos\text{AOC}-bc\cos\text{BOC}=mn\cos\text{ACB}-ab\cos\text{AOB},$$

which is the relation between the edges and their included angles.

Taking the vectors of (a), and squaring,

$$\left.\begin{aligned}[\mathrm{V}(\mathrm{CA}\,.\,\mathrm{CB})]^2 &= (\mathrm{V}\alpha\beta)^2 - \mathrm{V}\alpha\beta\mathrm{V}\alpha\gamma - \mathrm{V}\alpha\beta\mathrm{V}\gamma\beta - \mathrm{V}\alpha\gamma\mathrm{V}\alpha\beta \\ &\quad + (\mathrm{V}\alpha\gamma)^2 + \mathrm{V}\alpha\gamma\mathrm{V}\gamma\beta - \mathrm{V}\gamma\beta\mathrm{V}\alpha\beta + \mathrm{V}\gamma\beta\mathrm{V}\alpha\gamma + (\mathrm{V}\gamma\beta)^2.\end{aligned}\right\} \quad (b)$$

But

$$\begin{aligned}-(\mathrm{V}\alpha\beta\mathrm{V}\gamma\beta + \mathrm{V}\gamma\beta\mathrm{V}\alpha\beta) &= -2\mathrm{S}\,.\,\mathrm{V}\alpha\beta\mathrm{V}\gamma\beta \qquad \text{(Eq. 55)} \\ &= 2\mathrm{TV}\alpha\beta\mathrm{TV}\gamma\beta \cos \mathrm{B},\end{aligned}$$

in which B is the angle between the planes AOB, BOC.

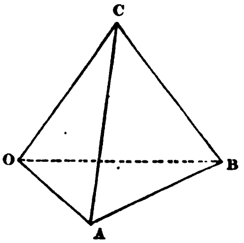
Also

$$-(\mathrm{V}\alpha\beta\mathrm{V}\alpha\gamma + \mathrm{V}\alpha\gamma\mathrm{V}\alpha\beta) = -2\mathrm{S}\,.\,\mathrm{V}\alpha\beta\mathrm{V}\alpha\gamma = 2\mathrm{TV}\alpha\beta\mathrm{TV}\alpha\gamma \cos \mathrm{A},$$

and

$$\begin{aligned}\mathrm{V}\alpha\gamma\mathrm{V}\gamma\beta + \mathrm{V}\gamma\beta\mathrm{V}\alpha\gamma = 2\mathrm{S}\,.\,\mathrm{V}\alpha\gamma\mathrm{V}\gamma\beta &= -2\mathrm{TV}\alpha\gamma\mathrm{TV}\gamma\beta \cos(180^\circ - \mathrm{C}) \\ &= 2\mathrm{TV}\alpha\gamma\mathrm{TV}\gamma\beta \cos \mathrm{C},\end{aligned}$$

Fig. 52.



in which A, B and C are the angles opposite the edges BC, AC and AB respectively. Hence (b) becomes

$$\begin{aligned}-[\mathrm{TV}(\mathrm{CA}\,.\,\mathrm{CB})]^2 = &-(\mathrm{TV}\alpha\beta)^2 - (\mathrm{TV}\alpha\gamma)^2 - (\mathrm{TV}\gamma\beta)^2 \\ &+ 2\mathrm{TV}\alpha\beta\mathrm{TV}\alpha\gamma \cos \mathrm{A} + 2\mathrm{TV}\alpha\beta\mathrm{TV}\gamma\beta \cos \mathrm{B} \\ &+ 2\mathrm{TV}\alpha\gamma\mathrm{TV}\gamma\beta \cos \mathrm{C}.\end{aligned}$$

But (Art. 41, 7th)

$$\mathrm{TV}(\mathrm{CA}\,.\,\mathrm{CB}) = 2 \text{ area ACB},$$

and similarly for the others. Hence, dividing by -4,

$$\begin{aligned}(\text{area ABC})^2 = &(\text{area AOB})^2 + (\text{area AOC})^2 + (\text{area BOC})^2 - \\ &2 \text{ area AOB area AOC} \cos \mathrm{A} - 2 \text{ area AOB area BOC} \cos \mathrm{B} - \\ &2 \text{ area AOC area BOC} \cos \mathrm{C},\end{aligned}$$

which is the relation between the plane faces and their included angles.

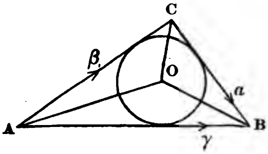
If the angles are right angles, then

$$(\text{area ABC})^2 = (\text{area AOB})^2 + (\text{area AOC})^2 + (\text{area BOC})^2.$$

24. *To inscribe a circle in a given triangle.*

Let α, β, γ (Fig. 53) be unit vectors along the sides. Then, Art. 16, the angle-bisectors are

Fig. 53.



$$\begin{aligned} &x(\beta+\gamma), \\ -&y(\gamma+\alpha), \\ &z(\alpha-\beta). \end{aligned}$$

Now

$$x(\beta+\gamma) = c\gamma - y(\gamma+\alpha).$$

Operating with $\text{V}\,.\,(\gamma+\alpha)\times$

$$x = \frac{c\text{V}\alpha\gamma}{\text{V}\gamma\beta + \text{V}\alpha\beta + \text{V}\alpha\gamma}.$$

Hence

$$\text{AO} = x(\beta+\gamma) = \frac{c\text{V}\alpha\gamma}{\text{V}\gamma\beta + \text{V}\alpha\beta + \text{V}\alpha\gamma}(\beta+\gamma),$$

or, since α, β, γ are unit vectors,

$$\text{AO} = \frac{c \sin \text{B}}{\sin \text{A} + \sin \text{B} + \sin \text{C}}(\beta+\gamma).$$

Squaring, to find the length of AO, we have, since $(\beta+\gamma)^2 = -2(1+\cos \text{A})$,

$$-\text{AO}^2 = -\left[\frac{c \sin \text{B}}{\sin \text{A} + \sin \text{B} + \sin \text{C}}\right]^2 2(1+\cos \text{A}),$$

$$\text{AO} = \frac{c \sin \text{B}}{\sin \text{A} + \sin \text{B} + \sin \text{C}}\sqrt{2(1+\cos \text{A})},$$

$$= \frac{c \sin \text{B}}{\sin \text{A} + \sin \text{B} + \sin \text{C}} 2 \cos \tfrac{1}{2}\text{A}.$$

25. *If tangents be drawn at the vertices of a triangle inscribed in a circle, their intersections with the opposite sides of the triangle will lie in a straight line.*

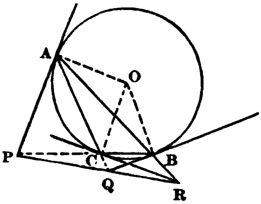
Let O be the center of the circle (Fig. 54) whose radius is r, and OA $= \alpha$, OB $= \beta$, OC $= \gamma$. Since OA and AP are at right angles,

$$S(\mathrm{OA} \cdot \mathrm{AP}) = 0.$$

But

$$\mathrm{AP} = \mathrm{AB} + \mathrm{BP} = \mathrm{AB} + y\mathrm{BC} = \beta - \alpha + y(\gamma - \beta);$$

Fig. 54.



hence, substituting this value above,

$$S\alpha[\beta - \alpha + y(\gamma - \beta)] = 0,$$
$$S\alpha^2 = S\alpha\beta + yS(\alpha\gamma - \alpha\beta),$$

and

$$y = -\frac{r^2 + S\alpha\beta}{S\alpha\gamma - S\alpha\beta};$$

Therefore

$$\mathrm{OP} = \mathrm{OB} + \mathrm{BP} = \beta + y\mathrm{BC} = \beta - \frac{r^2 + S\alpha\beta}{S\alpha\gamma - S\alpha\beta}(\gamma - \beta)$$
$$= \frac{(r^2 + S\alpha\gamma)\beta - (r^2 + S\alpha\beta)\gamma}{S\alpha\gamma - S\alpha\beta}.$$

Similarly, or, by a cyclic change of vectors,

$$\mathrm{OQ} = \frac{(r^2 + S\alpha\beta)\gamma - (r^2 + S\beta\gamma)\alpha}{S\alpha\beta - S\beta\gamma},$$
$$\mathrm{OR} = \frac{(r^2 + S\beta\gamma)\alpha - (r^2 + S\alpha\gamma)\beta}{S\beta\gamma - S\alpha\gamma}.$$

Whence

$$(S\alpha\gamma - S\alpha\beta)\mathrm{OP} + (S\alpha\beta - S\beta\gamma)\mathrm{OQ} + (S\beta\gamma - S\alpha\gamma)\mathrm{OR} = 0.$$

But also

$$(S\alpha\gamma - S\alpha\beta) + (S\alpha\beta - S\beta\gamma) + (S\beta\gamma - S\alpha\gamma) = 0.$$

Hence P, Q and R are collinear.

26. *The sum of the angles of a triangle is two right angles.*

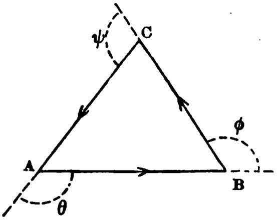
Let α, β, γ be unit vectors along BC, CA and AB (Fig. 55). Then (Art. 42)

$$\frac{\alpha}{\gamma} = \epsilon^{\frac{2\phi}{\pi}},$$

$$\frac{\beta}{\alpha} = \epsilon^{\frac{2\psi}{\pi}},$$

$$\frac{\gamma}{\beta} = \epsilon^{\frac{2\theta}{\pi}},$$

Fig. 55.



But

$$\frac{\alpha}{\gamma}\,\frac{\gamma}{\beta}\,\frac{\beta}{\alpha} = 1 = \epsilon^{\frac{2\phi}{\pi}}\,\epsilon^{\frac{2\theta}{\pi}}\,\epsilon^{\frac{2\psi}{\pi}} = \epsilon^{\frac{2}{\pi}(\phi+\theta+\psi)}.$$

Hence $\frac{2}{\pi}(\phi + \theta + \psi) =$ an even multiple of 2 (Art. 42), as $2n$, as we go round the triangle n times.

In taking the arithmetical sum, or passing once round, we take the first even multiple of 2, or

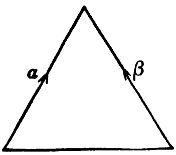
$$\frac{2}{\pi}(\phi + \theta + \psi) = 4;$$

$$\therefore\ \phi + \theta + \psi = 2\pi,$$

and the sum of the *interior* angles is $3\pi - 2\pi = \pi$, or two right angles.

27. *The angles at the base of an isosceles triangle are equal to each other.*

Fig. 56.



Let α and β (Fig. 56) be the vector sides of the triangle, and $T\alpha = T\beta$. Then, if the proposition be true,

$$\frac{\alpha}{\alpha - \beta} = K\frac{\beta}{\beta - \alpha},$$

or

$$\alpha(\alpha - \beta)^{-1} = K\beta(\beta - \alpha)^{-1} = (\beta - \alpha)^{-1}\beta,$$

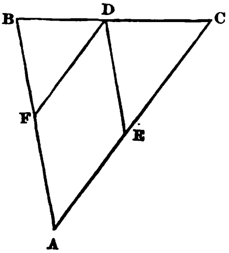
$$\alpha(\beta - \alpha) = (\alpha - \beta)\beta;$$

$$\therefore\ \alpha^2 = \beta^2,$$

which is true, since $T\alpha = T\beta$.

28. *To find a point on the base of a triangle such that, if lines be drawn through it parallel to and limited by the sides, they will be equal.*

Fig. 57.



Draw DE (Fig. 57) and DF parallel to the sides. From similar triangles, if $AE = xAC$,

$$x = \frac{AE}{AC} = \frac{BF}{BA} = \frac{BA - AF}{BA},$$

whence

$$1 - x = \frac{AF}{BA}.$$

Now

$$AD = AF + FD,$$

or, since $FD = AE$,

$$= (1-x)AB + xAC.$$

But, since FD is to be equal to ED,

$$(1-x)TAB = xTAC = y;$$
$$\therefore\ (1-x)TABUAB = yUAB,$$
$$xTACUAC = yUAC,$$

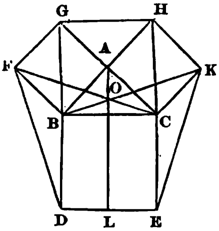
and therefore

$$AD = y(UAB + UAC),$$

and D is on the angle-bisector.

29. *If any line be drawn through the middle point of a line joining two parallels, it is bisected at that point.*

Fig. 58.



30. *If the diagonal of a parallelogram is an angle-bisector, the parallelogram is a rhombus.*

31. *In* any *triangle the sum of the squares of the lines* GH, KE, DF (Fig. 58) *is three times the sum of the squares of the sides of the triangle.*

32. *The sum of the angles about two right lines which intersect is four right angles.*

33. *If the sides of any polygon be produced so as to form one angle at each vertex, the sum of the angles is four right angles.*

34. *Find the eight roots of unity* (Art. 39).

35. *The square of the medial to any side of a triangle is one-half the sum of the squares of the sides which contain it, minus one-fourth the square of the third side.*

55. Product of two or more Vectors.

1. Let $q = \alpha\beta$, $r = \gamma$. Then, since $Sqr = Srq$,

$$S\alpha\beta\gamma = S\gamma\alpha\beta.$$

Let $q = \gamma\alpha$, $r = \beta$. Then

$$Sqr = Srq = S\gamma\alpha\beta = S\beta\gamma\alpha ;$$
$$\therefore\ S\alpha\beta\gamma = S\beta\gamma\alpha = S\gamma\alpha\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (108),$$

or, *the scalar of the product of three vectors is the same if the cyclical order is not changed.*

This may also be shown by means of the associative law of vector multiplication as follows:

$$\alpha\beta\gamma = (\alpha\beta)\gamma = (S\alpha\beta + V\alpha\beta)\gamma.$$

Taking the scalars

$$\begin{aligned} S\alpha\beta\gamma &= S(S\alpha\beta + V\alpha\beta)\gamma \\ &= S(V\alpha\beta \,.\, \gamma), \text{ since } S(S\alpha\beta \,.\, \gamma) = 0, \\ &= S \,.\, \gamma V\alpha\beta ; \end{aligned}$$

introducing the term $S \,.\, \gamma S\alpha\beta = 0$,

$$\begin{aligned} &= S \,.\, \gamma V\alpha\beta + S \,.\, \gamma S\alpha\beta \\ &= S \,.\, \gamma(S\alpha\beta + V\alpha\beta) \\ &= S\gamma(\alpha\beta) = S\gamma\alpha\beta. \end{aligned}$$

In a similar manner

$$\begin{aligned} S\alpha\beta\gamma &= S\,.\,\alpha(S\beta\gamma + V\beta\gamma) \\ &= S\,.\,\alpha V\beta\gamma \\ &= S(V\beta\gamma\,.\,\alpha) \\ &= S(V\beta\gamma + S\beta\gamma)\alpha \\ &= S\beta\gamma\alpha, \end{aligned}$$

and, as before,

$$S\alpha\beta\gamma = S\beta\gamma\alpha = S\gamma\alpha\beta.$$

2. Again

$$\begin{aligned} S\alpha\beta\gamma &= S\,.\,\alpha(S\beta\gamma + V\beta\gamma) \\ &= S\,.\,\alpha V\beta\gamma \\ &= -S\,.\,\alpha V\gamma\beta \\ &= -S\alpha(V\gamma\beta + S\gamma\beta); \\ \therefore\ S\alpha\beta\gamma &= -S\alpha\gamma\beta \quad . \ . \ . \ . \ . \ . \ . \quad (109), \end{aligned}$$

or, *a change in the cyclical order of three vectors changes the sign of the scalar of their product.*

3. Resuming

$$\alpha\beta\gamma = \alpha(\beta\gamma)$$

and taking the vectors,

$$\begin{aligned} V\alpha\beta\gamma &= V\,.\,\alpha(S\beta\gamma + V\beta\gamma) \\ &= \alpha S\beta\gamma + V\,.\,\alpha V\beta\gamma. \end{aligned}$$

Also

$$\begin{aligned} V\gamma\beta\alpha &= V(S\gamma\beta + V\gamma\beta)\alpha \\ &= V\,.\,\alpha S\gamma\beta - V\,.\,\alpha V\gamma\beta \\ &= V\,.\,\alpha S\gamma\beta + V\,.\,\alpha V\beta\gamma \\ &= V\,.\,\alpha(S\gamma\beta + V\beta\gamma) \\ &= \alpha S\beta\gamma + V\,.\,\alpha V\beta\gamma; \\ \therefore\ V\alpha\beta\gamma &= V\gamma\beta\alpha \quad . \ . \ . \ . \ . \ . \ . \ . \quad (110), \end{aligned}$$

or, *the vector of the product of three vectors is the same as the vector of their product in inverted order.*

4. *Geometrical interpretation of* $S\alpha\beta\gamma$.

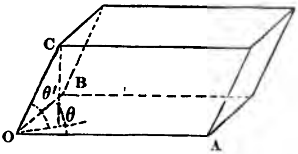
Let α, β, γ be unit vectors along the three adjacent edges OA, OB, OC (Fig. 59) of any parallelopiped, θ being the angle between α and β, and θ' the angle made by γ with the plane AOB. Then

$$\alpha\beta = -\cos\theta + \epsilon\sin\theta,$$

ϵ being a vector perpendicular to the plane AOB.

Operating with $\times\, \mathrm{S} \,.\, \gamma$

Fig. 59.



$$\begin{aligned} \mathrm{S}\alpha\beta\gamma &= \mathrm{S}(-\cos\theta + \epsilon\sin\theta)\gamma \\ &= \mathrm{S}(\sin\theta \,.\, \epsilon\gamma). \end{aligned}$$

But $\mathrm{S}\epsilon\gamma = -\cos$ of the angle between ϵ and $\gamma = -\sin\theta'$;

$$\therefore\ \mathrm{S}\alpha\beta\gamma = -\sin\theta\sin\theta'.$$

Now, if α, β, γ represent as vectors the edges OA, OB, OC, whose lengths are a, b, c,

$$\begin{aligned} \mathrm{S}\alpha\beta\gamma &= -\mathrm{T}\alpha\mathrm{T}\beta\mathrm{T}\gamma\sin\theta\sin\theta' \\ &= -abc\sin\theta\sin\theta'. \end{aligned}$$

But $ab\sin\theta =$ area of the parallelogram whose sides are a and b, and $c\sin\theta' =$ perpendicular from C on the plane AOB. Hence

$-\mathrm{S}\alpha\beta\gamma =$ *volume of a parallelopiped whose edges are* a, b *and* c, *drawn parallel to* α, β *and* γ.

Cor. 1. Whatever the order of the vectors, the volume is the same; hence, as already shown,

$$\pm\,\mathrm{S}\alpha\beta\gamma = \pm\,\mathrm{S}\beta\gamma\alpha = \pm\,\mathrm{S}\gamma\alpha\beta = \mp\,\mathrm{S}\alpha\gamma\beta, \text{ etc.}$$

Cor. 2. If $\mathrm{S}\alpha\beta\gamma = 0$, neither α, β or γ being zero, then either $\theta = 0$, or $\theta' = 0$, or *the vectors are complanar.*

Cor. 3. Conversely, if α, β, γ are complanar, $\mathrm{S}\alpha\beta\gamma = 0$.

Cor. 4. The volume of the triangular pyramid of which the edges are OC, OB, OA, is $-\frac{1}{6}\mathrm{S}\alpha\beta\gamma$.

5. We have seen that when α, β and γ are complanar, $\mathrm{S}\alpha\beta\gamma = 0$, and therefore $\alpha\beta\gamma$ is a vector. To find this vector, suppose a

triangle constructed whose sides AB, BC, CA have the directions of α, β and γ respectively, a vector not being changed by motion parallel to itself. Since the tensor of the vector sought is the product of the tensors of α, β and γ, we have to find $\mathrm{U}(\mathrm{AB} \,.\, \mathrm{BC} \,.\, \mathrm{CA})$, *i.e.*, its direction. Circumscribe on the triangle ABC a circle and draw a tangent at A, represented by T′AT. Since the angles TAB and BCA are equal, we have

$$\mathrm{U}\frac{\mathrm{BC}}{\mathrm{CA}} = \mathrm{U}\frac{\mathrm{AB}}{\mathrm{AT}'}\left[= \mathrm{U}\frac{\mathrm{BA}}{\mathrm{AT}}\right],$$

whence

$$\mathrm{U}(\mathrm{BC} \,.\, \mathrm{CA}) = \mathrm{U}(\mathrm{AB} \,.\, \mathrm{AT}')\left[= \mathrm{U}(\mathrm{BA} \,.\, \mathrm{AT})\right].$$

Introducing UAB ×

$$\mathrm{U}(\mathrm{AB} \,.\, \mathrm{BC} \,.\, \mathrm{CA}) = \mathrm{U}(\mathrm{AB} \,.\, \mathrm{AB} \,.\, \mathrm{AT}')\left[= \mathrm{U}(\mathrm{AB} \,.\, \mathrm{BA} \,.\, \mathrm{AT})\right],$$

or, since $\mathrm{U}(\mathrm{AB} \,.\, \mathrm{BA}) = -(\mathrm{U} \,.\, \mathrm{AB})^2 = 1,$

$$\mathrm{U}(\mathrm{AB} \,.\, \mathrm{BC} \,.\, \mathrm{CA}) = -\mathrm{U} \,.\, \mathrm{AT}' = \mathrm{U} \,.\, \mathrm{AT}.$$

Hence, if A, B, C are any three non-collinear points in a plane, or if α, β, γ are the sides of a triangle joining them, in order (in either direction, since $\mathrm{V}\alpha\beta\gamma = \mathrm{V}\gamma\beta\alpha$),

$$\alpha\beta\gamma, \quad \beta\gamma\alpha, \quad \gamma\alpha\beta$$

are the vector tangents to the circumscribing circle at the angles of the triangle.

Again, if A, B, C are any three points in a plane, not in a straight line, and α and β are two vectors along the two successive sides AB, BC of the triangle which they determine, and CD a vector drawn from C parallel to γ, intersecting the circumscribed circle at D, then is DA parallel to $\mathrm{V}\alpha\beta\gamma = \delta$. For

$$\delta = \alpha\beta\gamma = \alpha\frac{\beta}{\beta}\beta\gamma = \alpha\beta^2\beta^{-1}\gamma = -(\mathrm{T}\beta)^2\alpha\beta^{-1}\gamma = -(\mathrm{T}\beta)^2\frac{\alpha}{\beta}\gamma,$$

whence $\mathrm{U} \,.\, \frac{-\alpha}{\beta}$, which turns β parallel to $-\alpha$, turns γ into a direction $\delta = \mathrm{DA}$, the opposite angles of an inscribed quadrilateral being supplementary.

If γ have a direction such that CD crosses AB, or the quadrilateral is a crossed one, it is evident on construction of the figure that

$$\mathrm{U}\delta' = \mathrm{U}\alpha\beta\gamma = \mathrm{U}(\mathrm{AD}) = -\mathrm{U}\delta.$$

Hence the continued product of the three successive vector sides of a quadrilateral inscribed in a circle is parallel to the fourth side, its direction being towards or from the initial point as the quadrilateral is uncrossed or crossed; and, conversely, no plane quadrilateral can satisfy the above formula $\pm\mathrm{U}\delta = \mathrm{U}\alpha\beta\gamma$, unless A, B, C and D are con-circular. The continued product of the four successive sides of an inscribed quadrilateral is a scalar, for

$$\alpha\beta\gamma\delta = (\alpha\beta\gamma)\delta = \pm\delta^2 = \mp d^2.$$

Since the product of two vectors is a quaternion whose axis is perpendicular to their plane, while the product of a quaternion by a vector perpendicular to its axis is another vector perpendicular to its axis, and so on, it follows that the continued product of any even number of complanar vectors is generally a quaternion whose axis is perpendicular to their plane, while the product of any odd number of complanar vectors is a vector in the same plane. Hence the formulae

$$\mathrm{S}\alpha = 0, \quad \mathrm{S}\alpha\beta\gamma = 0, \quad \mathrm{S}\alpha\beta\gamma\delta\sigma = 0, \quad \text{etc.},$$

for complanar vectors.

If, however, the given vectors are parallel to the sides of a polygon ABC ····· MN inscribed in a circle, then

$$\mathrm{U}(\mathrm{AB}\cdot\mathrm{BC}\cdot\mathrm{CD}\cdots\mathrm{MN}\cdot\mathrm{NA}) = \mathrm{U}(\mathrm{AB}\cdot\mathrm{BC}\cdot\mathrm{CA})\,\mathrm{U}(\mathrm{AC}\cdot\mathrm{CD}\cdot\mathrm{DA})\cdots \times \mathrm{U}(\mathrm{AM}\cdot\mathrm{MN}\cdot\mathrm{NA}).$$

But each of the products $\mathrm{U}(\mathrm{AB}\cdot\mathrm{BC}\cdot\mathrm{CA})$ is equal to $\mathrm{U}\cdot\mathrm{AT}$, AT being the tangent to the circle at A. Hence

$$\mathrm{U}(\mathrm{AB}\cdot\mathrm{BC}\cdot\mathrm{CD}\cdots\mathrm{MN}\cdot\mathrm{NA}) = (\mathrm{U}\cdot\mathrm{AT})^n,$$

which reduces, according as n is even or odd, to ± 1 or $\pm\mathrm{U}\cdot\mathrm{AT}$. Hence the product of the vectors will be a scalar or a vector according as their number is even or odd, and in the latter case this vector is parallel to the tangent at A.

If the vectors are not complanar, but parallel to the successive sides of a gauche polygon inscribed in a sphere, the polygon may be divided as above into triangles, for each of which the product of the three successive sides is a vector tangent to the circumscribing circle, all these vectors lying in the tangent plane to the sphere at the initial point. If the number of sides is even, their product will be a quaternion whose axis is perpendicular to the tangent plane, *i.e.*, lies in the direction of the radius of the sphere to the initial point; if odd, the product is a vector in the tangent plane.

Hence, if A, B, C and D are four given points, not in a plane, $AB = \alpha$, $BC = \beta$, $CD = \gamma$ being given vectors, and P any other point such that $DP = \sigma$, $PA = \rho$, if P lies on the surface of a sphere through the four given points, we have the necessary and sufficient condition

$$\alpha\beta\gamma\sigma\rho = \rho\sigma\gamma\beta\alpha,$$

for each member is equal to minus the conjugate of the other, and must therefore (Art. 46) be a vector.

6. From Equation (56),

$$\beta\gamma - \gamma\beta = 2\,\mathbf{V}\beta\gamma.$$

Operating with $\mathbf{V}\,.\,\alpha\times$

$$2\mathbf{V}\,.\,\alpha\mathbf{V}\beta\gamma = \mathbf{V}\,.\,\alpha(\beta\gamma - \gamma\beta).$$

Introducing in the second member $\beta\alpha\gamma - \beta\alpha\gamma$,

$$\begin{aligned} &= \mathbf{V}\,(\alpha\beta\gamma - \alpha\gamma\beta + \beta\alpha\gamma - \beta\alpha\gamma) \\ &= \mathbf{V}(\alpha\beta + \beta\alpha)\gamma - \mathbf{V}(\alpha\gamma\beta + \gamma\alpha\beta) \\ &= \mathbf{V}\,.\,2(\mathbf{S}\alpha\beta)\gamma - \mathbf{V}(\alpha\gamma + \gamma\alpha)\beta \\ &= 2\,\gamma\mathbf{S}\alpha\beta - 2\,\beta\mathbf{S}\alpha\gamma. \end{aligned}$$

Hence

$$\mathbf{V}\,.\,\alpha\mathbf{V}\beta\gamma = \gamma\mathbf{S}\alpha\beta - \beta\mathbf{S}\alpha\gamma \quad . \quad . \quad . \quad . \quad (111).$$

This formula may be extended. Thus, for α write $\mathrm{V}\alpha\delta$, and we have

$$\mathrm{V}\,.\,\mathrm{V}\alpha\delta\mathrm{V}\beta\gamma = \gamma\mathrm{S}(\mathrm{V}\alpha\delta)\beta - \beta\mathrm{S}(\mathrm{V}\alpha\delta)\gamma,$$

$$\mathrm{V}\,.\,\mathrm{V}\alpha\delta\mathrm{V}\beta\gamma = \gamma\mathrm{S}\alpha\delta\beta - \beta\mathrm{S}\alpha\delta\gamma \quad . \quad . \quad . \quad . \quad (112).$$

An inspection of this formula shows that it gives a vector complanar with γ and β. Moreover, since

$$\mathrm{V}\,.\,\mathrm{V}\alpha\delta\mathrm{V}\beta\gamma = \mathrm{V}\,.\,\mathrm{V}\gamma\beta\mathrm{V}\alpha\delta = \delta\mathrm{S}\gamma\beta\alpha - \alpha\mathrm{S}\gamma\beta\delta,$$

it is also complanar with α and δ, and is, therefore, parallel to the line of intersection of the planes of α, δ, and β, γ.

Similarly

$$\mathrm{V}\,.\,\mathrm{V}\beta\gamma\mathrm{V}\alpha\delta = \delta\mathrm{S}\beta\gamma\alpha - \alpha\mathrm{S}\beta\gamma\delta = -\mathrm{V}\,.\,\mathrm{V}\alpha\delta\mathrm{V}\beta\gamma \quad . \quad (113).$$

Adding Equations (112) and (113)

$$\delta\mathrm{S}\beta\gamma\alpha - \alpha\mathrm{S}\beta\gamma\delta + \gamma\mathrm{S}\alpha\delta\beta - \beta\mathrm{S}\alpha\delta\gamma = 0 \quad . \quad . \quad (114),$$

or

$$\delta\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma\delta - \beta\mathrm{S}\alpha\gamma\delta + \gamma\mathrm{S}\alpha\beta\delta \quad . \quad . \quad . \quad (115),$$

a formula expressing a vector δ in terms of any three given diplanar vectors, α, β, γ; so that, if

$$\mathrm{S}\beta\gamma\delta = b, \quad -\mathrm{S}\alpha\gamma\delta = \mathrm{S}\gamma\alpha\delta = c, \quad \mathrm{S}\alpha\beta\delta = a, \quad \mathrm{S}\alpha\beta\gamma = m,$$

$$\delta = m^{-1}(b\alpha + c\beta + a\gamma).$$

7. Resuming Equation (111), and adding $\alpha\mathrm{S}\beta\gamma$ to both members,

$$\mathrm{V}\,.\,\alpha\mathrm{V}\beta\gamma + \alpha\mathrm{S}\beta\gamma = \gamma\mathrm{S}\alpha\beta - \beta\mathrm{S}\alpha\gamma + \alpha\mathrm{S}\beta\gamma,$$

whence

$$\mathrm{V}\,.\,\alpha(\mathrm{S}\beta\gamma + \mathrm{V}\beta\gamma) =$$

$$\mathrm{V}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma - \beta\mathrm{S}\alpha\gamma + \gamma\mathrm{S}\alpha\beta \quad . \quad . \quad . \quad . \quad (116).$$

The form of this equation shows that α and γ may be interchanged, or that $\mathrm{V}\alpha\beta\gamma = \mathrm{V}\gamma\beta\alpha$, as already shown.

Again, replacing α by $\mathrm{V}\alpha\beta$ in Equation (111),

$$\mathrm{V}\,.\,\mathrm{V}\alpha\beta\mathrm{V}\beta\gamma = \gamma\mathrm{S}(\mathrm{V}\alpha\beta)\beta - \beta\mathrm{S}(\mathrm{V}\alpha\beta)\gamma,$$

or

$$\mathrm{V}\,.\,\mathrm{V}\alpha\beta\mathrm{V}\beta\gamma = -\beta\mathrm{S}\alpha\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

8. Writing $V\gamma\delta\alpha\beta$ first as $V(\gamma \,.\, \delta\alpha\beta)$, and then as $V(\gamma\delta \,.\, \alpha\beta)$, we have

$$\begin{aligned} V(\gamma \,.\, \delta\alpha\beta) &= V \,.\, \gamma(S\delta\alpha\beta + V\delta\alpha\beta) \\ &= \gamma S\delta\alpha\beta + V \,.\, \gamma V\delta\alpha\beta \\ \text{[Equation (116)]} \quad &= \gamma S\alpha\beta\delta + V\gamma\delta S\alpha\beta - V\gamma\alpha S\delta\beta + V\gamma\beta S\delta\alpha. \qquad (a) \\ V(\gamma\delta \,.\, \alpha\beta) &= V(S\gamma\delta + V\gamma\delta)(S\alpha\beta + V\alpha\beta) \\ &= V\gamma\delta S\alpha\beta + V\alpha\beta S\gamma\delta + V \,.\, V\gamma\delta V\alpha\beta \\ &= V\gamma\delta S\alpha\beta + V\alpha\beta S\gamma\delta - V \,.\, V\alpha\beta V\gamma\delta, \end{aligned}$$

or, Equation (112),

$$= V\gamma\delta S\alpha\beta + V\alpha\beta S\gamma\delta - \delta S\alpha\beta\gamma + \gamma S\alpha\beta\delta. \qquad (b)$$

Equating (*a*) and (*b*),

$$\delta S\alpha\beta\gamma = V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta + V\alpha\beta S\gamma\delta \quad . \quad . \quad (118),$$

a formula expressing a vector δ in terms of three other vectors resulting from their products taken two and two; so that, if $S\alpha\beta\gamma = m$, $S\alpha\delta = a$, $S\beta\delta = b$, $S\gamma\delta = c$,

$$\delta = m^{-1}(aV\beta\gamma + bV\gamma\alpha + cV\alpha\beta).$$

Operating on Equation (118) with $S \,.\, \rho\, \times$, we obtain, since $S \,.\, \rho V\gamma\alpha = S\rho\gamma\alpha$,

$$S\rho\delta S\alpha\beta\gamma - S\beta\delta S\rho\gamma\alpha - S\alpha\delta S\rho\beta\gamma - S\gamma\delta S\rho\alpha\beta = 0,$$

or

$$S\alpha\delta S\rho\beta\gamma - S\beta\delta S\gamma\rho\alpha + S\gamma\delta S\rho\alpha\beta - S\rho\delta S\alpha\beta\gamma = 0 \quad . \quad (119),$$

a formula eliminating δ.

56. Exercises.

Prove the following relations:

1. $S\alpha\beta\gamma\delta = S\delta\alpha\beta\gamma$.
2. $\alpha\beta \,.\, \beta\gamma = -\alpha\gamma$.
3. $\alpha^2\beta^2 = \alpha\beta \,.\, \beta\alpha$.
4. $S \,.\, V\alpha\beta V\beta\gamma = S \,.\, \alpha\beta V\beta\gamma$ . . . . (120).
5. $S\alpha\beta\gamma\delta = S\alpha\beta S\gamma\delta - S\alpha\gamma S\beta\delta + S\alpha\delta S\beta\gamma$ . . . . . . (121), from which show that $S\alpha\beta\gamma\delta = S\beta\gamma\delta\alpha$.

6. $S\,.\,V\alpha\beta V\gamma\delta = S\alpha\delta S\beta\gamma - S\alpha\gamma S\beta\delta$ . . . . (122).
7. $S(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = 2\,S\alpha\beta\gamma$.
8. $\alpha\beta\gamma + \gamma\beta\alpha = 2\,V\alpha\beta\gamma$.
9. $\alpha\beta\gamma - \gamma\beta\alpha = 2\,S\alpha\beta\gamma$.
10. $V(\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta) = 0$ . . . . (123).
11. $V\alpha\beta\gamma + V\gamma\alpha\beta = 2\,\gamma S\alpha\beta$.
12. $\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3\,S\alpha\beta\gamma$.
13. $S\,.\,V\alpha\beta V\beta\gamma V\gamma\alpha = -\,(S\alpha\beta\gamma)^2$.
14. $S\,.\,\gamma V\beta\alpha = \gamma\beta\alpha - \gamma S\beta\alpha + \beta S\gamma\alpha - \alpha S\beta\gamma$ . . . . (124).
15. $S\,.\,V(V\alpha\beta V\beta\gamma)\,V(V\beta\gamma V\gamma\alpha)\,V(V\gamma\alpha V\alpha\beta) = -\,(S\alpha\beta\gamma)^4$.
16. $S[V\alpha\beta V\gamma\delta + V\alpha\gamma V\delta\beta + V\alpha\delta V\beta\gamma] = 0$ . . . . . (125).
17. If $S\alpha\beta\gamma = m$, $S\alpha\rho = 0$, $S\beta\rho = 0$, $S\gamma\rho = 0$, show that $\rho = 0$. Conversely, if ρ is not zero, then $S\alpha\beta\gamma = 0$.
18. Interpret $\rho = \alpha^{-1}\beta\alpha$.

We have first, directly,

$$T\rho = T\beta,$$
$$S\alpha\rho\beta = S\alpha\alpha^{-1}\beta\alpha\beta = S\beta\alpha\beta = S\beta^2\alpha = 0\,;$$

$\therefore\ \rho$, α and β are complanar.

$$S\alpha\rho = S\alpha\alpha^{-1}\beta\alpha = S\beta\alpha,$$
$$-\,T\rho T\alpha\cos\theta = -\,T\alpha T\beta\cos\phi,$$

or, since $T\rho = T\beta$, $\cos\theta = \cos\phi$.

Similarly $V\alpha\rho = V\beta\alpha$, and $\sin\theta = \sin\phi$. Hence

$$\theta = \phi,$$

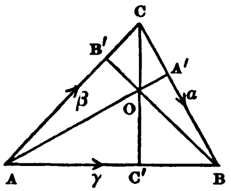
and α bisects the angle between β and ρ.

19. Show that $\rho = \alpha\beta\alpha^{-1} = \alpha^{-1}(S\alpha\beta - V\alpha\beta)$.
20. ρ being any vector, show that $V\,.\,V\alpha\rho V\rho\beta = x\rho$.
21. If $S\alpha\beta = -\alpha^2$, show that α is perpendicular to $\beta - \alpha$.
22. What are the relative directions of α and β, if $K\frac{\beta}{\alpha} = -\frac{\beta}{\alpha}$? If $K\frac{\beta}{\alpha} = \frac{\beta}{\alpha}$?

57. Examples.

1. *The altitudes of a triangle intersect in a point.*

Fig. 60.



Let (Fig. 60) AC $= \beta$, CB $= \alpha$, AB $= \gamma$. Then vectors along C'C, B'B and A'A are

$$\epsilon\gamma, \quad -\epsilon\beta, \quad -\epsilon\alpha$$

respectively. Now

$$\text{AO} = \text{AC} + \text{CO} = \text{AB} + \text{BO},$$

or

$$\beta - x\epsilon\gamma = \gamma + y\epsilon\beta.$$

Operating with $\times$ S . β, we have, since $yS\epsilon\beta^2 = 0$,

$$x = -\frac{S\alpha\beta}{S\epsilon\gamma\beta}.$$

Having assumed O to be the intersection of the altitudes BB' and CC', let O' be the intersection of AA' and CC'. Then

$$\text{AO}' = \text{AC} + \text{CO}',$$

or

$$z\epsilon\alpha = \beta - x'\epsilon\gamma.$$

Operating with $\times$ S . α

$$\begin{aligned} x' &= \frac{S\beta\alpha}{S\epsilon\gamma\alpha} = \frac{S\beta\alpha}{S\alpha\epsilon\gamma} \\ &= \frac{S\beta\alpha}{S(\gamma - \beta)\epsilon\gamma} = \frac{S\alpha\beta}{-S\beta\epsilon\gamma} \\ &= -\frac{S\gamma\beta}{S\epsilon\gamma\beta}. \end{aligned}$$

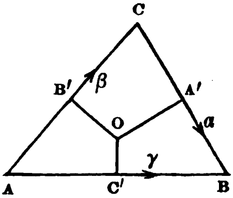
Hence O and O' coincide, and

$$\text{AO} = \beta + \frac{S\gamma\beta}{S\epsilon\gamma\beta}\epsilon\gamma.$$

2. *To circumscribe a circle about a triangle.*

Let (Fig. 61) $AC = \beta$, $CB = \alpha$, $AB = \gamma$.

Fig. 61.



Then

$$A'O = -x\epsilon\alpha,$$
$$C'O = y\epsilon\gamma,$$
$$B'O = -z\epsilon\beta.$$

Operating with $\times S . \beta$ on the expression

$$AO = \tfrac{1}{2}\gamma + y\epsilon\gamma = \tfrac{1}{2}\beta - z\epsilon\beta,$$

we have

$$y = -\frac{S\alpha\beta}{2\,S\epsilon\gamma\beta}.$$

Operating with $\times S . \alpha$ on

$$BO' = -\tfrac{1}{2}\gamma + y'\epsilon\gamma = -\tfrac{1}{2}\alpha - x'\epsilon\alpha,$$

we have

$$y' = \frac{S\alpha\beta}{2\,S\epsilon\gamma\alpha} = -\frac{S\alpha\beta}{2\,S\epsilon\gamma\beta}.$$

Therefore $y = y'$ and O and O′ coincide.

The radius may be found by squaring

$$AO = \tfrac{1}{2}\gamma + y\epsilon\gamma = \tfrac{1}{2}\gamma - \frac{S\alpha\beta}{2\,S\epsilon\gamma\beta}\epsilon\gamma,$$

whence

$$-R^2 = -\frac{c^2}{4} - \frac{c^2a^2b^2\cos^2 C}{4\,b^2c^2\sin^2 A},$$

since, if a, b, c are the tensors of α, β, γ,

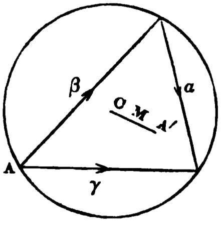
$$\frac{\gamma^2}{4} = -\frac{c^2}{4},$$
$$S(\tfrac{1}{2}\gamma \,.\, y\epsilon\gamma) = 0,$$
$$y^2(\epsilon\gamma)^2 = -c^2\frac{a^2b^2\cos^2 C}{4\,b^2c^2\sin^2 A}.$$

Hence

$$R = \frac{\sqrt{c^2\sin^2 A + a^2\cos^2 C}}{2\sin A} = \frac{a}{2\sin A}.$$

3. *In any triangle, the centre of the circumscribed circle, the intersection of the altitudes and the intersection of the medials lie in the same straight line; and the distance between the last two points is two-thirds of the distance between the first two.*

Fig. 62.



Let M (Fig. 62) be the intersection of the medials, A′ that of the altitudes, and C the center of the circle.

Then, from Ex. 5, Art. 11, where CP (Fig. 11) is given in terms of the adjacent sides, we have

$$\text{AM} = \tfrac{1}{3}(\beta + \gamma).$$

From Ex. 1, Art. 57,

$$\text{AA}' = \beta + \frac{S\alpha\beta}{S\epsilon\gamma\beta}\epsilon\gamma.$$

From Ex. 2, Art. 57,

$$\text{AC} = \tfrac{1}{2}\left(\gamma - \frac{S\alpha\beta}{S\epsilon\gamma\beta}\epsilon\gamma\right).$$

But

$$\text{CM} = \text{AM} - \text{AC} = \tfrac{1}{3}\beta - \tfrac{1}{6}\gamma + \tfrac{1}{2}\frac{S\alpha\beta}{S\epsilon\gamma\beta}\epsilon\gamma,$$

and

$$\text{MA}' = \text{AA}' - \text{AM} = \tfrac{2}{3}\beta - \tfrac{1}{3}\gamma + \frac{S\alpha\beta}{S\epsilon\gamma\beta}\epsilon\gamma.$$

$$\therefore\ \text{MA}' = 2\,\text{CM},$$

and, since, as vectors, they are multiples of each other, and have a common point, they form one and the same straight line.

4. *To find the condition that the perpendiculars from the angles of a tetraedron to the opposite faces shall intersect.*

With the notation of Fig. 52, the perpendiculars from A and B on the opposite faces are

$$V\beta\gamma \quad \text{and} \quad V\gamma\alpha.$$

If they intersect, at P say, then must A, B, P lie in one plane. Hence, Art. 55, 4, Cor. 3,

$$S[(\beta - \alpha)V\beta\gamma V\gamma\alpha] = 0,$$

or

$$S(\beta-\alpha)[S\,.\,V\beta\gamma V\gamma\alpha + V\,.\,V\beta\gamma V\gamma\alpha] = 0,$$
$$S(\beta-\alpha)V\,.\,V\beta\gamma V\gamma\alpha = 0.$$

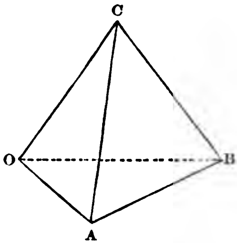


Fig. 52 (*bis*).

But, Equation (117),

$$V\,.\,V\beta\gamma V\gamma\alpha = -\gamma S\beta\gamma\alpha\,;$$
$$\therefore\ -(S\beta\gamma - S\alpha\gamma)S\beta\gamma\alpha = 0,$$

or

$$S\beta\gamma = S\alpha\gamma. \qquad (a)$$

From the figure, we have

$$\begin{aligned} \mathrm{BC}^2 + \mathrm{OA}^2 &= (\gamma-\beta)^2 + \alpha^2 \\ &= \gamma^2 - 2\,S\gamma\beta + \beta^2 + \alpha^2 \\ \text{or, from } (a), \qquad &= \gamma^2 - 2\,S\alpha\gamma + \beta^2 + \alpha^2 \\ &= (\gamma-\alpha)^2 + \beta^2 \\ &= \mathrm{AC}^2 + \mathrm{OB}^2. \end{aligned}$$

Hence the condition is that *the sums of the squares of each pair of opposite edges shall be the same.*

5. Interpret Equation (118),

$$\delta S\alpha\beta\gamma = V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta + V\alpha\beta S\gamma\delta,$$

under the condition that α, β, γ be complanar with δ.

If α, β, γ are complanar, $S\alpha\beta\gamma = 0$, and therefore, δ being in or out of the plane,

$$S\alpha\delta V\beta\gamma + S\beta\delta V\gamma\alpha + S\gamma\delta V\alpha\beta = 0. \qquad (a)$$

If δ be in the plane, we have for any four co-initial lines OA, OB, OC, OD,

$$\sin \mathrm{BOC} \cos \mathrm{AOD} + \sin \mathrm{COA} \cos \mathrm{BOD} + \sin \mathrm{AOB} \cos \mathrm{COD} = 0,$$

and, for a line perpendicular to OD,

$$\sin \mathrm{BOC} \sin \mathrm{AOD} + \sin \mathrm{COA} \sin \mathrm{BOD} + \sin \mathrm{AOB} \sin \mathrm{COD} = 0.$$

If δ is perpendicular to the plane, the terms in (a) vanish separately.

6. *If X, Y, Z be the angles made by any line* OP *with three rectangular axes, then*

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1.$$

From Equation (67)

$$i\rho = xi^2 + yij + zik = -x + yk - zj,$$

whence

$$(Si\rho)^2 = x^2.$$

Operating in a similar manner with $S.j\times$ and $S.k\times$ we obtain

$$-\rho^2 = (Si\rho)^2 + (Sj\rho)^2 + (Sk\rho)^2.$$

If $T\rho = r$, then $\rho^2 = -r^2$, $Si\rho = -r\cos X$, etc. Hence

$$\text{OP}^2 = \text{OP}^2(\cos^2 X + \cos^2 Y + \cos^2 Z),$$

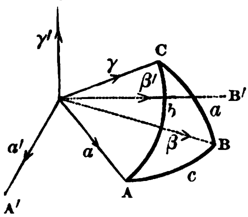
or

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1.$$

Applications to Spherical Trigonometry.

Let ABC (Fig. 63) be any spherical triangle on the surface of a unit sphere whose center is O; α, β, γ being unit vectors from O to the vertices. The sides AB, BC, CA represent versors whose angles are c, a, b, and axes are $\text{OC}' = \gamma'$, $\text{OA}' = \alpha'$, $\text{OB}' = \beta'$; α', β', γ' being unit vectors to the vertices of the polar triangle whose sides are a', b', c', the supplements of the opposite angles A, B, C of the triangle ABC.

Fig. 63.



7. We have first

$$\frac{\beta}{\gamma} = \frac{\beta}{\alpha}\cdot\frac{\alpha}{\gamma}. \qquad (a)$$

Taking the scalars, we have [Equation (90)],

$$\mathrm{S}\frac{\beta}{\gamma} = \mathrm{S}\frac{\beta}{\alpha}\,\mathrm{S}\frac{\alpha}{\gamma} + \mathrm{S}\,.\,\mathrm{V}\frac{\beta}{\alpha}\,\mathrm{V}\frac{\alpha}{\gamma}.$$

But

$$\mathrm{S}\frac{\beta}{\gamma} = \cos a, \quad \mathrm{S}\frac{\beta}{\alpha} = \cos c, \quad \mathrm{S}\frac{\alpha}{\gamma} = \cos b,$$

and

$$\begin{aligned}\mathrm{S}\,.\,\mathrm{V}\frac{\beta}{\alpha}\mathrm{V}\frac{\alpha}{\gamma} &= \mathrm{S}(\sin c\,.\,\gamma')(\sin b\,.\,\beta')\\ &= \sin c \sin b\, \mathrm{S}\gamma'\beta'\\ &= -\sin c \sin b \cos a'\\ &= \sin c \sin b \cos \mathrm{A}.\end{aligned}$$

Hence, in (a),

$$\cos a = \cos c \cos b + \sin c \sin b \cos \mathrm{A}.$$

By a cyclic permutation of the letters in (a), we obtain

$$\frac{\gamma}{\alpha} = \frac{\gamma}{\beta}\,\frac{\beta}{\alpha}. \qquad (b)$$

Whence, as before

$$\mathrm{S}\frac{\gamma}{\alpha} = \mathrm{S}\frac{\gamma}{\beta}\mathrm{S}\frac{\beta}{\alpha} + \mathrm{S}\,.\,\mathrm{V}\frac{\gamma}{\beta}\mathrm{V}\frac{\beta}{\alpha},$$

or

$$\cos b = \cos a \cos c + \sin a \sin c\, \mathrm{S}\alpha'\gamma',$$

in which $\mathrm{S}\alpha'\gamma' = -\cos b' = \cos \mathrm{B}.$

$$\therefore \cos b = \cos a \cos c + \sin a \sin c \cos \mathrm{B}. \qquad (c)$$

Similarly, or directly by cyclic permutation in (c),

$$\cos c = \cos b \cos a + \sin b \sin a \cos \mathrm{C}.$$

From the relation

$$\frac{\beta'}{\gamma'} = \frac{\beta'}{\alpha'}\,\frac{\alpha'}{\gamma'}$$

may be deduced in like manner

$$-\cos \mathrm{A} = \cos \mathrm{C} \cos \mathrm{B} - \sin \mathrm{C} \sin \mathrm{B} \cos a.$$

8. Resuming the equation

$$\frac{\beta}{\gamma}=\frac{\beta}{\alpha}\,\frac{\alpha}{\gamma}$$

of the last example, and taking the vectors, we have [Equation (91)],

$$\mathrm{V}\frac{\beta}{\gamma}=\mathrm{S}\frac{\beta}{\alpha}\mathrm{V}\frac{\alpha}{\gamma}+\mathrm{S}\frac{\alpha}{\gamma}\mathrm{V}\frac{\beta}{\alpha}+\mathrm{V}\,.\,\mathrm{V}\frac{\beta}{\alpha}\mathrm{V}\frac{\alpha}{\gamma}. \qquad (a)$$

But

$$\mathrm{V}\frac{\beta}{\gamma}=-\alpha'\sin a,$$

$$\mathrm{S}\frac{\beta}{\alpha}\mathrm{V}\frac{\alpha}{\gamma}=\cos c\,(\beta'\sin b)=\cos c\sin b\,.\,\beta',$$

$$\mathrm{S}\frac{\alpha}{\gamma}\mathrm{V}\frac{\beta}{\alpha}=\cos b\,(\gamma'\sin c)=\cos b\sin c\,.\,\gamma',$$

$$\mathrm{V}\,.\,\mathrm{V}\frac{\beta}{\alpha}\mathrm{V}\frac{\alpha}{\gamma}=\mathrm{V}(\gamma'\sin c)(\beta'\sin b)=\sin c\sin b\,\mathrm{V}\gamma'\beta'$$

$$=\sin c\sin b(-\alpha\sin a')=-\sin c\sin b\sin \mathrm{A}\,.\,\alpha.$$

Substituting in (a),

$$-\sin a\,.\,\alpha'=\cos c\sin b\,.\,\beta'+\cos b\sin c\,.\,\gamma'-\sin c\sin b\sin\mathrm{A}\,.\,\alpha. \qquad (b)$$

Operating with $\times\, \mathrm{S}\,.\,\gamma'^{-1}$,

$$-\sin a\,.\,\mathrm{S}\frac{\alpha'}{\gamma'}=\cos c\sin b\,\mathrm{S}\frac{\beta'}{\gamma'}+\cos b\sin c\,\mathrm{S}\frac{\gamma'}{\gamma'}-\sin c\sin b\sin\mathrm{A}\,\mathrm{S}\frac{\alpha}{\gamma'},$$

in which

$$\mathrm{S}\frac{\alpha'}{\gamma'}=\cos b'=-\cos\mathrm{B},$$

$$\mathrm{S}\frac{\beta'}{\gamma'}=-\cos\mathrm{A},$$

$$\mathrm{S}\frac{\alpha}{\gamma'}=0,$$ since α and γ' are at right angles.

Hence

$$\sin a\cos\mathrm{B}=\cos b\sin c-\cos c\sin b\cos\mathrm{A},$$

and in the same manner, or by a cyclic permutation of the letters,

$$\sin b \cos \text{C} = \cos c \sin a - \cos a \sin c \cos \text{B},$$
$$\sin c \cos \text{A} = \cos a \sin b - \cos b \sin a \cos \text{C}.$$

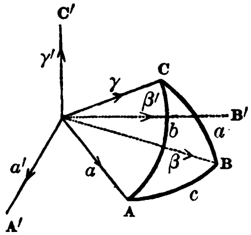
9. Operating on Equation (b) of the last example with $\times \mathbf{V} \cdot \gamma'^{-1}$ instead of $\times \mathbf{S} \cdot \gamma'^{-1}$,

$$-\sin a \, \mathbf{V}\frac{\alpha'}{\gamma'} = \cos c \sin b \, \mathbf{V}\frac{\beta'}{\gamma'} + \cos b \sin c \, \mathbf{V}\frac{\gamma'}{\gamma'} - \sin c \sin b \sin \text{A} \, \mathbf{V}\frac{\alpha}{\gamma'}.$$

But

$$\mathbf{V}\frac{\alpha'}{\gamma'} = \beta \sin b' = \beta \sin \text{B},$$
$$\mathbf{V}\frac{\beta'}{\gamma'} = -\alpha \sin a' = -\alpha \sin \text{A},$$
$$\mathbf{V}\frac{\gamma'}{\gamma'} = 0.$$

Fig. 63.



Substituting these values

$$-\sin a \sin \text{B} \,.\, \beta = -\cos c \sin b \sin \text{A} \,.\, \alpha$$
$$\qquad - \sin c \sin b \sin \text{A} \,.\, \mathbf{V}\frac{\alpha}{\gamma}.$$

Operating with $\times \alpha^{-1}$, and substituting for

$$\frac{\beta}{\alpha} = \cos c + \gamma' \sin c,$$

we obtain

$$-\sin a \sin \text{B} \cos c - \sin a \sin \text{B} \sin c \,.\, \gamma' = -\cos c \sin b \sin \text{A}$$
$$\qquad - \sin c \sin b \sin \text{A} \,.\, \gamma'.$$

Equating the scalar or vector parts, we have in either case

$$\sin a \sin \text{B} = \sin \text{A} \sin b,$$

or

$$\sin a : \sin b :: \sin \text{A} : \sin \text{B}.$$

The formulae of the preceding examples have all been deduced from the equation $\frac{\beta}{\gamma} = \frac{\beta}{\alpha}\,\frac{\alpha}{\gamma}$. The product as well as the quotient may also be employed, as follows:

10. Assuming the vector product

$$V\alpha\beta V\beta\gamma,$$

and taking the vector part, we have [Equation (117)],

$$V\,.\,V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma. \qquad (a)$$

But

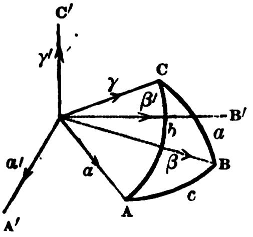
$$V\,.\,V\alpha\beta V\beta\gamma = V(\gamma'\sin c)(\alpha'\sin a) = \sin c \sin a \sin \mathrm{B}\,.\,\beta,$$

and, Art. 55, 4,

$$S\alpha\beta\gamma = -\sin c \sin\theta',$$

θ' being the angle made by OC with the plane of c. Substituting in (a),

Fig. 63.



$$\sin c \sin a \sin \mathrm{B}\,.\,\beta = \sin c \sin\theta'\,.\,\beta,$$

or

$$\sin\theta' = \sin a \sin \mathrm{B}.$$

By permutation, from (a),

$$V\,.\,V\gamma\alpha V\alpha\beta = -\alpha S\gamma\alpha\beta = -\alpha S\alpha\beta\gamma,$$

or

$$\sin b \sin c \sin \mathrm{A}\,.\,\alpha = \sin c \sin\theta'\,.\,\alpha,$$

$$\therefore\ \sin\theta' = \sin b \sin \mathrm{A}.$$

Equating these values of $\sin\theta'$, we have, as in Example 9,

$$\sin a : \sin b :: \sin \mathrm{A} : \sin \mathrm{B}.$$

11. Let p_a, p_b, p_c represent the arcs drawn from the vertices of ABC perpendicular to the opposite sides.

Resuming Equation (a) of the preceding example, and taking the tensors,

$$\begin{aligned} TV\,.\,V\alpha\beta V\beta\gamma = S\alpha\beta\gamma &= \sin c \sin p_c, \\ = S\beta\gamma\alpha &= \sin a \sin p_a, \\ = S\gamma\alpha\beta &= \sin b \sin p_b, \end{aligned}$$

and, taking the tensor of $\mathrm{V}\,.\,\mathrm{V}\alpha\beta\mathrm{V}\beta\gamma$ from the last example,

$$\sin c \sin a \sin \mathrm{B} = \sin a \sin p_a = \sin b \sin p_b = \sin c \sin p_c,$$

or

$$\sin p_a = \sin c \sin \mathrm{B},$$
$$\sin p_b = \frac{\sin c \sin a}{\sin b} \sin \mathrm{B},$$
$$\sin p_c = \sin a \sin \mathrm{B}.$$

12. *Show that if* ABC, A′B′C′ *be two tri-rectangular triangles on the surface of a sphere,*

$$\cos \mathrm{AA}' = \cos \mathrm{BB}' \cos \mathrm{CC}' - \cos \mathrm{B}'\mathrm{C} \cos \mathrm{BC}',$$

the triangles being lettered in the same order.

Let α, β, γ, α', β', γ' be the vectors to the vertices. These being at right angles, in each triangle, we have

$$\cos \mathrm{AA}' = -\,\mathrm{S}\alpha\alpha' = -\,\mathrm{S}\,.\,\mathrm{V}\beta\gamma\mathrm{V}\beta'\gamma',$$

or, Equation (122),

$$\begin{aligned}\cos \mathrm{AA}' &= \mathrm{S}\beta\beta'\mathrm{S}\gamma\gamma' - \mathrm{S}\beta'\gamma\mathrm{S}\beta\gamma' \\ &= \cos \mathrm{BB}' \cos \mathrm{CC}' - \cos \mathrm{B}'\mathrm{C} \cos \mathrm{BC}'.\end{aligned}$$

[The vectors of Equation (122) are arbitrary, but we may divide both members by the tensor of the product of the vectors, so that

$$\mathrm{S}(\mathrm{VU}\alpha\beta\mathrm{VU}\gamma\delta) = \mathrm{SU}\alpha\delta\mathrm{SU}\beta\gamma - \mathrm{SU}\alpha\gamma\mathrm{SU}\beta\delta,$$

for the unit sphere.]

13. Let ABCD be a spherical quadrilateral whose sides are AB $= a$, BC $= b$, CD $= c$, DA $= d$, the vectors to the poles of these arcs being α', β', γ', δ' respectively. Then

$$\mathrm{V}\alpha\beta = \alpha' \sin a,$$
$$\mathrm{V}\gamma\delta = \gamma' \sin c.$$

From Equation (122),

$$S \,.\, V\alpha\beta V\gamma\delta = S\alpha\delta S\beta\gamma - S\alpha\gamma S\beta\delta,$$

or

$$\sin a \sin c \, S\alpha'\gamma' = (-\cos \text{DA})(-\cos \text{BC}) - (-\cos \text{DB})(-\cos \text{AC}).$$

But $S\alpha'\gamma' = -\cos \text{L}$, L being the angle formed by the arcs AB and CD where they meet, the arcs being estimated in the directions indicated by the order of their terminal letters. Hence

$$\sin \text{AB} \sin \text{CD} \cos \text{L} = \cos \text{AC} \cos \text{BD} - \cos \text{AD} \cos \text{BC},$$

a formula due to Gauss.

14. Retaining the above notation, ABCD being still a spherical quadrilateral, denote the angles at the intersections of the arcs AB and CD, AC and DB, AD and BC, by L, M and N respectively. Then, from Equation (125),

$$S[V\alpha\beta V\gamma\delta + V\alpha\gamma V\delta\beta + V\alpha\delta V\beta\gamma] = 0,$$

we have identically

$$\sin \text{AB} \sin \text{CD} \cos \text{L} + \sin \text{AC} \sin \text{BD} \cos \text{M} + \sin \text{AD} \sin \text{BC} \cos \text{N} = 0.$$

Were the points A, B, C, D on the same great circle, the angles L, M and N would be zero, and the above reduces to

$$\sin \text{AB} \sin \text{CD} + \sin \text{AC} \sin \text{BD} + \sin \text{AD} \sin \text{BC} = 0,$$

and for a line OA′ perpendicular to OA and in the same plane, dropping the accent, we have

$$\cos \text{AB} \sin \text{CD} + \cos \text{AC} \sin \text{BD} + \cos \text{AD} \sin \text{BC} = 0,$$

which are the results of Example 5 of this article.

58. General Formulae.

1. We have seen, Equation (86), that $\mathbf{S}\Sigma = \Sigma\mathbf{S}$ and $\mathbf{V}\Sigma = \Sigma\mathbf{V}$; but (Art. 50, 4) that $\Sigma\mathbf{T}$ is *not* equal to $\mathbf{T}\Sigma$, nor $\Sigma\mathbf{U}$ to $\mathbf{U}\Sigma$. We have also seen, Equations (96) and (97), that $\mathbf{T}\Pi = \Pi\mathbf{T}$ and $\mathbf{U}\Pi = \Pi\mathbf{U}$; but $\mathbf{S}\Pi$ is *not* equal to $\Pi\mathbf{S}$, nor $\mathbf{V}\Pi$ to $\Pi\mathbf{V}$: for, 1st, $\mathbf{S}\Pi$ is independent of the factors under the Π sign, provided the product remains the same, while $\Pi\mathbf{S}$ is dependent upon them; and, 2d, because (Art. 55, 5) $\Pi\mathbf{V}$ is not necessarily a vector.

2. Resuming Equation (92),

$$\mathbf{S}rq = \mathbf{S}qr,$$

and, since r is arbitrary, writing rs for r, we have, by the associative law (Art. 52),

$$\begin{aligned} \mathbf{S}(rs)q &= \mathbf{S}q(rs), \\ \mathbf{S}r(sq) &= \mathbf{S}(sq)r, \\ \therefore\ \mathbf{S}rsq &= \mathbf{S}sqr = \mathbf{S}qrs \quad . \quad . \quad . \quad . \quad (126), \end{aligned}$$

a formula which may evidently be extended. Hence, *the scalar of the product of any number of quaternions is the same, so long as the cyclical order is maintained.*

3. Let p, q, r, s be four quaternions, such that

$$qr = ps. \qquad (a)$$

Operating with $\mathbf{K}q\times$,

$$\mathbf{K}q \,.\, qr = (\mathbf{K}q \,.\, q)r = (q\mathbf{K}q)r = \mathbf{K}q \,.\, ps,$$

since conjugate quaternions are commutative. Hence

$$(\mathbf{T}q)^2 r = \mathbf{K}q \,.\, ps,$$

or

$$r = \frac{\mathbf{K}q \,.\, ps}{(\mathbf{T}q)^2} = \mathbf{R}q \,.\, ps = \frac{1}{q} \cdot ps \quad . \quad . \quad . \quad (127).$$

Operating on (a) with $\times\mathbf{K}r$, we have

$$qr \,.\, \mathbf{K}r = ps \,.\, \mathbf{K}r,$$

or

$$q(\mathrm{T}r)^2 = ps\mathrm{K}r,$$

$$\therefore\ q = \frac{ps\mathrm{K}r}{(\mathrm{T}r)^2} = ps\mathrm{R}r = ps\frac{1}{r} \quad . \quad . \quad . \quad (128).$$

Hence, *in any equation of the products of two quaternions, the first factor of one member may be removed by writing its conjugate as the first factor of the second member, and dividing the latter by the square of the tensor, or simply by introducing the reciprocal as the first factor in the second member.* By substituting the word *last* for *first*, the above rule will apply to the second transformation.

4. Resuming, for facility of reference, the equations

$$q = \frac{\alpha}{\beta} = \frac{\mathrm{T}\alpha}{\mathrm{T}\beta}(\cos\phi + \epsilon\sin\phi) = \mathrm{T}q\,.\,\mathrm{U}q = \mathrm{S}q + \mathrm{V}q, \qquad (A)$$

$$q^{-1} = \frac{1}{q} = \frac{\beta}{\alpha} = \frac{\mathrm{T}\beta}{\mathrm{T}\alpha}(\cos\phi - \epsilon\sin\phi), \qquad (B)$$

$$\mathrm{K}q = \frac{\mathrm{T}\alpha}{\mathrm{T}\beta}(\cos\phi - \epsilon\sin\phi) = \mathrm{S}q - \mathrm{V}q, \qquad (C)$$

we observe directly that

$$\mathrm{S}q = \mathrm{S}(\mathrm{T}q\,.\,\mathrm{U}q) = \mathrm{T}q\,.\,\mathrm{SU}q \quad . \quad . \quad . \quad (129),$$

$$\mathrm{V}q = \mathrm{TV}q\,.\,\mathrm{UV}q = \mathrm{T}q\,.\,\mathrm{VU}q \quad . \quad . \quad . \quad (130),$$

$$\mathrm{TV}q = \mathrm{T}q\,.\,\mathrm{TVU}q = \mathrm{TVK}q \quad . \quad . \quad . \quad . \quad (131).$$

5. It has been already shown (Art. 54, Fig. 40) that $(\mathrm{T}\alpha)^2 + (\mathrm{T}\beta)^2 = (\mathrm{T}\gamma)^2$, and (Art. 54, Fig. 42) that $\mathrm{T}\alpha = \mathrm{T}\gamma\,.\cos\phi$, $\mathrm{T}\beta = \mathrm{T}\gamma\,.\sin\phi$; and therefore

$$(\mathrm{T}\gamma)^2\cos^2\phi + (\mathrm{T}\gamma)^2\sin^2\phi = (\mathrm{T}\gamma)^2,$$

or

$$\sin^2\phi + \cos^2\phi = 1.$$

Hence, from Equations (44),

$$(\mathrm{SU}q)^2 + (\mathrm{TVU}q)^2 = 1 \quad . \quad . \quad . \quad . \quad (132).$$

This important formula might have been written at once by assuming the above well-known relation of Plane Trigonometry.

6. From Equations (129) and (131), we may write Equation (132) under the form

$$(\mathbf{S}q)^2+(\mathbf{TV}q)^2=(\mathbf{T}q)^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (133),$$

or, from Equation (107),

$$(\mathbf{S}q)^2-(\mathbf{V}q)^2=(\mathbf{T}q)^2=(\mathbf{S}q)^2+(\mathbf{TV}q)^2 \quad . \quad (134),$$

since $\epsilon^2=-1$.

7. Comparing (A), (B) and (C),

$$\mathbf{SU}q=\mathbf{SU}\frac{1}{q}=\mathbf{SUK}q \quad . \; . \; . \quad (135),$$

$$\mathbf{TVU}q=\mathbf{TVU}\frac{1}{q}=\mathbf{TVUK}q \quad . \; . \quad (136),$$

and from Equations (129) and (135),

$$\mathbf{S}q=\mathbf{T}q \,.\, \mathbf{SU}q=\mathbf{T}q \,.\, \mathbf{SU}\frac{1}{q}=\mathbf{T}q \,.\, \mathbf{SUK}q \,. \quad . \quad (137).$$

8. Since $\mathbf{T}q=\mathbf{TK}q$, we have

$$\mathbf{T}q \,.\, \mathbf{TK}q=(\mathbf{T}q)^2 \quad . \; . \; . \; . \; . \quad (138),$$

and $\mathbf{T}q$ being a positive scalar,

$$\mathbf{KT}q=\mathbf{TK}q \quad . \; . \; . \; . \; . \quad (139).$$

As exercises in the transformation of these and the following symbolical equations, some of the results already obtained will be deduced anew. Thus, to prove that $\mathbf{T}(qq')=\mathbf{T}q\mathbf{T}q'$, whence $\mathbf{T} \,.\, q^2=(\mathbf{T}q)^2$, we have

$$\begin{aligned}(\mathbf{T}qq')^2 &= (qq')\mathbf{K}(qq') && \text{Equation (107)}\\ &= qq'\mathbf{K}q'\mathbf{K}q && \text{Equation (99)}\\ &= q(q'\mathbf{K}q')\mathbf{K}q=(\mathbf{T}q')^2 q\mathbf{K}q \\ &= (\mathbf{T}q')^2(\mathbf{T}q)^2,\\ \therefore \mathbf{T}qq' &= \mathbf{T}q\mathbf{T}q'.\end{aligned}$$

9. Substituting for $\mathbf{S}q$ and $\mathbf{TV}q$ their values from Equations (79) and (131)

$$(\mathbf{SK}q)^2+(\mathbf{TVK}q)^2=(\mathbf{S}q)^2+(\mathbf{TV}q)^2 \quad . \quad . \quad (140).$$

10. Resuming from Art. 51, 1, the expressions

$$\begin{aligned} \mathbf{V}rq &= \mathbf{S}r\mathbf{V}q+\mathbf{S}q\mathbf{V}r+\mathbf{V}\,.\,\mathbf{V}r\mathbf{V}q, && (a)\\ \mathbf{V}qr &= \mathbf{S}q\mathbf{V}r+\mathbf{S}r\mathbf{V}q+\mathbf{V}\,.\,\mathbf{V}q\mathbf{V}r, && (b)\\ \mathbf{S}qr &= \mathbf{S}q\mathbf{S}r+\mathbf{S}\,.\,\mathbf{V}q\mathbf{V}r, && (c)\end{aligned}$$

we have, by adding and subtracting,

$$\left.\begin{aligned} \mathbf{V}qr+\mathbf{V}rq &= 2\,\mathbf{S}q\mathbf{V}r+2\,\mathbf{S}r\mathbf{V}q\\ \mathbf{V}qr-\mathbf{V}rq &= 2\,\mathbf{V}\,.\,\mathbf{V}q\mathbf{V}r\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (141).$$

And, if $q=r$, from (a) and (c),

$$\left.\begin{aligned} \mathbf{V}\,.\,q^2 &= 2\,\mathbf{S}q\mathbf{V}q\\ \mathbf{S}\,.\,q^2 &= (\mathbf{S}q)^2+(\mathbf{V}q)^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (142),$$

whence

$$q^2=(\mathbf{S}q)^2+2\,\mathbf{S}q\mathbf{V}q+(\mathbf{V}q)^2 \quad . \quad . \quad . \quad (143).$$

Dividing Equations (142) by $(\mathbf{T}q)^2$

$$\left.\begin{aligned} \mathbf{SU}\,.\,q^2 &= (\mathbf{SU}q)^2+(\mathbf{VU}q)^2\\ \mathbf{VU}\,.\,q^2 &= 2\,\mathbf{SU}q\,.\,\mathbf{VU}q\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (144),$$

since, evidently,

$$\left.\begin{aligned} \mathbf{S}\,.\,q^2 &= (\mathbf{T}q)^2\mathbf{SU}\,.\,q^2\\ \mathbf{V}\,.\,q^2 &= (\mathbf{T}q)^2\mathbf{VU}\,.\,q^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (145).$$

Again, substituting in the second of Equations (142) the value of $(\mathbf{V}q)^2$ from Equation (134), we have

$$\mathbf{S}\,.\,q^2=2(\mathbf{S}q)^2-(\mathbf{T}q)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (146),$$

and dividing by $(\mathbf{T}q)^2$

$$\mathbf{SU}\,.\,q^2=2(\mathbf{SU}q)^2-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147).$$

Substituting $(\mathbf{S}q)^2$ from the same equation

$$\mathbf{S}\,.\,q^2=2(\mathbf{V}q)^2+(\mathbf{T}q)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (148).$$

Equations (146) and (148) may be written

$$(\mathbf{S}+\mathbf{T})q^2=2(\mathbf{S}q)^2 \quad \text{and} \quad (\mathbf{S}-\mathbf{T})q^2=2(\mathbf{V}q)^2.$$

Introducing in (a), or (b), the condition that q and r are complanar, we have, after substituting versors,

$$\mathbf{VU}qr=\mathbf{VU}q\mathbf{SU}r+\mathbf{VU}r\mathbf{SU}q,$$

since, under the condition, $\mathbf{V}(\mathbf{VU}q\mathbf{VU}r)=0$.

Taking the tensors, since q and r are complanar,

$$\mathbf{TVU}qr=\mathbf{TVU}q\mathbf{SU}r+\mathbf{SU}q\mathbf{TVU}r \quad . \quad . \quad . \quad (149),$$

and, interpreting, Art. 51, 6,

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

Introducing the same condition of complanarity in (c)

$$\mathbf{S}qr=\mathbf{S}q\mathbf{S}r-\mathbf{TV}q\mathbf{TV}r,$$

or, substituting versors as above,

$$\mathbf{SU}qr=\mathbf{SU}q\mathbf{SU}r-\mathbf{TVU}q\mathbf{TVU}r \quad . \quad . \quad . \quad (150),$$

or, interpreting,

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\phi\sin\theta.$$

11. Putting Equation (146) under the form

$$\mathbf{S}q=\sqrt{\frac{\mathbf{S}\,.\,q^2+\mathbf{T}\,.\,q^2}{2}},$$

and writing $\sqrt{q}$ for q, we have

$$\mathbf{S}\sqrt{q}=\sqrt{\tfrac{1}{2}(\mathbf{S}q+\mathbf{T}q)} \quad . \quad . \quad . \quad . \quad (151).$$

12. Taking the tensors of the first of Equations (142), we have

$$\mathbf{TV}q=\frac{\mathbf{TV}\,.\,q^2}{2\,\mathbf{S}q},$$

and writing $\sqrt{q}$ for q

$$\mathbf{TV} \cdot \sqrt{q} = \frac{\mathbf{TV}q}{2\mathbf{S}\sqrt{q}},$$

or, by Equations (133) and (151),

$$\mathbf{TV} \cdot \sqrt{q} = \tfrac{1}{2}\sqrt{\frac{(\mathbf{T}q)^2 - (\mathbf{S}q)^2}{\frac{1}{2}(\mathbf{T}q + \mathbf{S}q)}},$$

whence

$$\mathbf{TV} \cdot \sqrt{q} = \sqrt{\tfrac{1}{2}(\mathbf{T}q - \mathbf{S}q)} \quad . \quad . \quad . \quad . \quad . \quad (152),$$

and

$$(\mathbf{TV} : \mathbf{S})\sqrt{q} = \sqrt{\frac{\mathbf{T}q - \mathbf{S}q}{\mathbf{T}q + \mathbf{S}q}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (153).$$

13. From the definition of the powers of a quaternion, we have

$$q^{-m}q^m = 1, \quad \left(q^{\frac{m}{n}}\right)^n = q^m \quad . \quad . \quad . \quad . \quad . \quad (154).$$

Hence, since $q = \mathbf{T}q \cdot \mathbf{U}q$, $\mathbf{T}\Pi = \Pi\mathbf{T}$ and $\mathbf{U}\Pi = \Pi\mathbf{U}$,

$$\mathbf{T}q^{-m} \cdot \mathbf{T}q^m = 1, \quad \mathbf{U}q^{-m} \cdot \mathbf{U}q^m = 1 \quad . \quad . \quad (155).$$

Also, because $\mathbf{U}q^{-m} = \mathbf{UK}q^m$,

$$q^{-m} = \mathbf{T}q^{-m} \cdot \mathbf{U}q^{-m} = \mathbf{T}q^{-m} \cdot \mathbf{UK}q^m = \mathbf{T}q^{-2m}\mathbf{K}q^m,$$

or, since $\mathbf{K}pq = \mathbf{K}q\mathbf{K}p$, writing pq for q, and making $m = 1$,

$$\begin{aligned}(pq)^{-1} &= \mathbf{T}(pq)^{-2}\mathbf{K}pq = \mathbf{T}(pq)^{-2}\mathbf{K}q\mathbf{K}p \\ &= \mathbf{T}(pq)^{-2}(\mathbf{T}q)^2(\mathbf{T}p)^2q^{-1}p^{-1}.\end{aligned}$$

Or

$$(pq)^{-1} = q^{-1}p^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (156),$$

the reciprocal of the product of two quaternions being equal to the product of their reciprocals in inverted order.

This formula may be extended by the Associative principle, by a process similar to that employed in the deduction of Equation (126), so that if Π' represent the product of the same factors as those of Π, in reverse order,

$$(\Pi q)^{-1} = \Pi' q^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (157).$$

The equation $\mathbf{K}pq = \mathbf{K}q\mathbf{K}p$ may be deduced without reference to spherical arcs. For, by Art. 44, any two quaternions can be reduced to the forms $q = \frac{\beta}{a}$, $p = \frac{\gamma}{\beta}$, whence

$$pq = \frac{\gamma}{a}, \quad \text{or} \quad pq \cdot a = \gamma, \quad p\beta = \gamma,$$

and therefore

$$\mathbf{K}p \cdot \gamma = \mathbf{K}p \cdot p\beta = (\mathbf{K}p \cdot p)\beta = (\mathbf{T}p)^2\beta.$$

Now

$$\begin{aligned} (\mathbf{K}q\mathbf{K}p)\gamma &= \mathbf{K}q(\mathbf{T}p)^2\beta = (\mathbf{T}p)^2\mathbf{K}q \cdot \beta \\ &= (\mathbf{T}p)^2\mathbf{K}q \cdot qa = (\mathbf{T}p)^2(\mathbf{T}q)^2 a = (\mathbf{T}pq)^2 a \\ &= \mathbf{K}pq \cdot pq \cdot a \;= \mathbf{K}pq \cdot \gamma \\ &\therefore\ \mathbf{K}pq = \mathbf{K}q\mathbf{K}p, \end{aligned}$$

which, by the Associative law, gives

$$\mathbf{K}\Pi = \Pi'\mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158).$$

14. Show that $\mathbf{K}(-q) = -\mathbf{K}q$.

15. Show that

$$\begin{aligned} \mathbf{T}(p+q)^2 &= (p+q)(\mathbf{K}p + \mathbf{K}q) \\ &= (\mathbf{T}p)^2 + (\mathbf{T}q)^2 + 2\,\mathbf{S} \cdot p\mathbf{K}q \\ &= (\mathbf{T}p)^2 + (\mathbf{T}q)^2 + 2\,\mathbf{T}p\mathbf{T}q\mathbf{SU} \cdot p\mathbf{K}q \\ &= (\mathbf{T}p + \mathbf{T}q)^2 - 2\,\mathbf{T}p\mathbf{T}q(1 - \mathbf{SU} \cdot p\mathbf{K}q), \end{aligned}$$

and therefore that $\mathbf{T}(p+q)$ cannot be greater than the sum or less than the difference of $\mathbf{T}p$ and $\mathbf{T}q$.

16. Show that $q\mathbf{UV}q^{-1} = \mathbf{TV}q - \mathbf{S}q\mathbf{UV}q$.

59. Applications to Plane Trigonometry.

1. For formulae involving 2θ, let

$$q = \mathbf{T}q(\cos 2\theta + \epsilon \sin 2\theta).$$

Then

$$\sqrt{q} = q' = \sqrt{\mathbf{T}q}(\cos\theta + \epsilon\sin\theta).$$

From Equation (142), $\mathbf{S} \,.\, q^2 = (\mathbf{S}q)^2 + (\mathbf{V}q)^2$, we then have

$$\mathbf{S}q = (\mathbf{S}q')^2 + (\mathbf{V}q')^2,$$

or, dividing out $\mathbf{T}q$,

$$\mathbf{SU}q = (\mathbf{SU}q')^2 + (\mathbf{VU}q')^2;$$

and, interpreting,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Again, from Equation (147), $\mathbf{SU} \,.\, q^2 = 2(\mathbf{SU}q)^2 - 1$,

$$\mathbf{SU}q = 2(\mathbf{SU}q')^2 - 1;$$

whence

$$\cos 2\theta = 2\cos^2\theta - 1.$$

Again, from Equation (142), $\mathbf{V} \,.\, q^2 = 2\,\mathbf{S}q\mathbf{V}q$,

$$\mathbf{V}q = 2\,\mathbf{S}q'\mathbf{V}q',$$

or, dividing out $\mathbf{T}q$ and ϵ,

$$\mathbf{TVU}q = 2\,\mathbf{SU}q'\mathbf{TVU}q';$$

whence

$$\sin 2\theta = 2\cos\theta \sin\theta.$$

2. Resuming Equations (149) and (150),

$$\mathbf{TVU}qr = \mathbf{TVU}q\mathbf{SU}r + \mathbf{SU}q\mathbf{TVU}r,$$
$$\mathbf{SU}qr = \mathbf{SU}q\mathbf{SU}r - \mathbf{TVU}q\mathbf{TVU}r,$$

which have already been interpreted as the sine and cosine of the sum of two angles, and writing for

$$r = \mathbf{T}r(\cos\phi + \epsilon\sin\phi), \quad r^{-1} = \frac{1}{\mathbf{T}r}(\cos\phi - \epsilon\sin\phi),$$

q and r being complanar, we have

$$\mathbf{TVU}qr^{-1} = \mathbf{TVU}q\mathbf{SU}r - \mathbf{SU}q\mathbf{TVU}r \quad . \quad . \quad (159),$$
$$\mathbf{SU}qr^{-1} = \mathbf{SU}q\mathbf{SU}r + \mathbf{TVU}q\mathbf{TVU}r \quad . \quad . \quad (160),$$

or, interpreting,

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta,$$
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

3. Adding Equations (149) and (159),

$$\mathbf{TVU}qr + \mathbf{TVU}qr^{-1} = 2\,\mathbf{SU}r\mathbf{TVU}q,$$

in which, if $qr = p$, $qr^{-1} = t$, $\therefore\ q = \sqrt{pt}$, $r = \sqrt{pt^{-1}}$ (Art. 58, 3),

$$\mathbf{TVU}p + \mathbf{TVU}t = 2\,\mathbf{SU}(\sqrt{pt^{-1}})\mathbf{TVU}(\sqrt{pt}) \quad . \quad (161),$$

or

$$\sin x + \sin y = 2\cos\tfrac{1}{2}(x-y)\sin\tfrac{1}{2}(x+y).$$

Similarly, by subtracting the same equations,

$$\mathbf{TVU}qr - \mathbf{TVU}qr^{-1} = 2\,\mathbf{SU}q\mathbf{TVU}r,$$
$$\mathbf{TVU}p - \mathbf{TVU}t = 2\,\mathbf{SU}(\sqrt{pt})\mathbf{TVU}(\sqrt{pt^{-1}}) \quad . \quad (162),$$

or

$$\sin x - \sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).$$

4. From Equations (150) and (160), by addition and subtraction, we obtain, in a similar manner,

$$\mathbf{SU}p + \mathbf{SU}t = 2\,\mathbf{SU}(\sqrt{pt})\mathbf{SU}(\sqrt{pt^{-1}}) \quad . \quad . \quad . \quad (163),$$

and

$$\mathbf{SU}p - \mathbf{SU}t = -2\,\mathbf{TVU}(\sqrt{pt})\mathbf{TVU}(\sqrt{pt^{-1}}),$$

whence

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y),$$
$$\cos y - \cos x = 2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).$$

5. Resuming Equation (152),

$$\mathbf{TV}\sqrt{q} = \sqrt{\tfrac{1}{2}(\mathbf{T}q - \mathbf{S}q)},$$

it may be put under the form

$$2(\mathbf{TVU}\sqrt{q})^2 = 1 - \mathbf{SU}q,$$

or

$$2\sin^2\tfrac{1}{2}\theta = 1 - \cos\theta.$$

and, in a similar manner, from Equation (151),

$$\mathbf{S}\sqrt{q} = \sqrt{\tfrac{1}{2}(\mathbf{S}q + \mathbf{T}q)},$$
$$2(\mathbf{SU}\sqrt{q})^2 = \mathbf{SU}q + 1,$$

or

$$2\cos^2\tfrac{1}{2}\theta = 1 + \cos\theta.$$

6. From Equation (142)

$$(\mathbf{TV} : \mathbf{S})q^2 = \frac{2\,\mathbf{S}q\mathbf{TV}q}{(\mathbf{S}q)^2 + (\mathbf{V}q)^2}$$

$$= \frac{2\,\mathbf{TV}q}{\mathbf{S}q} \cdot \frac{(\mathbf{S}q)^2}{(\mathbf{S}q)^2 - (\mathbf{TV}q)^2}$$

$$= \frac{2(\mathbf{TV} : \mathbf{S})q}{1 - [(\mathbf{TV} : \mathbf{S})q]^2},$$

or

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

And, in a similar manner,

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}.$$

7. From Equations (90) and (91), q and r being complanar,

$$\mathbf{S}qr = \mathbf{S}q\mathbf{S}r + \mathbf{S}\,.\,\mathbf{V}q\mathbf{V}r = \mathbf{S}q\mathbf{S}r - \mathbf{TV}q\mathbf{TV}r,$$
$$\mathbf{TV}qr = \mathbf{S}q\mathbf{TV}r + \mathbf{S}r\mathbf{TV}q,$$

we have, by division,

$$(\mathbf{TV} : \mathbf{S})qr = \frac{\mathbf{S}q\mathbf{TV}r + \mathbf{S}r\mathbf{TV}q}{\mathbf{S}q\mathbf{S}r - \mathbf{TV}q\mathbf{TV}r}$$

$$= \frac{(\mathbf{TV} : \mathbf{S})r + (\mathbf{TV} : \mathbf{S})q}{1 - (\mathbf{TV} : \mathbf{S})q(\mathbf{TV} : \mathbf{S})r},$$

or

$$\tan(\theta + \phi) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}.$$

Also

$$(\mathbf{TV} : \mathbf{S})qr^{-1} = \frac{(\mathbf{TV} : \mathbf{S})q - (\mathbf{TV} : \mathbf{S})r}{1 + (\mathbf{TV} : \mathbf{S})q(\mathbf{TV} : \mathbf{S})r},$$

or

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}.$$

8. From Equation (153)

$$(\mathbf{TV} : \mathbf{S})\sqrt{q} = \sqrt{\frac{\mathbf{T}q - \mathbf{S}q}{\mathbf{T}q + \mathbf{S}q}},$$

by substituting in succession p and t for $\sqrt{q}$, $\therefore\ q = p^2$ and $q = t^2$, we obtain, after reduction,

$$(\mathbf{TV} : \mathbf{S})p + (\mathbf{TV} : \mathbf{S})t = \frac{\mathbf{TV}p\mathbf{S}t + \mathbf{TV}t\mathbf{S}p}{\mathbf{S}p\mathbf{S}t},$$

or, dividing out $\mathbf{T}p\mathbf{T}t$ from the second member, and applying Equation (149),

$$(\mathbf{TV} : \mathbf{S})p + (\mathbf{TV} : \mathbf{S})t = \frac{\mathbf{TVU}pt}{\mathbf{SU}p\mathbf{SU}t},$$

or

$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}.$$

And similarly, by subtraction and applying Equation (159),

$$(\mathbf{TV} : \mathbf{S})p - (\mathbf{TV} : \mathbf{S})t = \frac{\mathbf{TVU}pt^{-1}}{\mathbf{SU}p\mathbf{SU}t},$$

or

$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x \cos y}.$$

9. From Equations (161) and (163)

$$\mathbf{TVU}\sqrt{pt} = \frac{\mathbf{TVU}p + \mathbf{TVU}t}{2\,\mathbf{SU}\sqrt{pt^{-1}}},$$

$$\mathbf{SU}\sqrt{pt} = \frac{\mathbf{SU}p + \mathbf{SU}t}{2\,\mathbf{SU}\sqrt{pt^{-1}}},$$

whence

$$(\mathbf{TVU} : \mathbf{SU})\sqrt{pt} = (\mathbf{TV} : \mathbf{S})\sqrt{pt} = \frac{\mathbf{TVU}p + \mathbf{TVU}t}{\mathbf{SU}p + \mathbf{SU}t},$$

or

$$\tan\tfrac{1}{2}(x+y) = \frac{\sin x + \sin y}{\cos x + \cos y}.$$

And, in a similar manner, from Equations (162) and (163),

$$(\mathbf{TV} : \mathbf{S})\sqrt{pt^{-1}} = \frac{\mathbf{TVU}p - \mathbf{TVU}t}{\mathbf{SU}p + \mathbf{SU}t},$$

or

$$\tan\tfrac{1}{2}(x-y) = \frac{\sin x - \sin y}{\cos x + \cos y}.$$

10. Similar formulae may be deduced for functions of other ratios of an angle. Thus, from Equation (90), writing rs for r, and making $q = r = s$ all complanar, we have, by Equation (142),

$$\mathbf{S} \,.\, q^3 = (\mathbf{S}q)^3 - 3\,\mathbf{S}q(\mathbf{TV}q)^2,$$

or

$$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta,$$

or, under the more familiar form,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

CHAPTER III.

Applications to Loci.

60. Any vector, as ρ, may be resolved into three component vectors parallel to any three given vectors, as α, β, γ, no two of which are parallel, and which are not parallel to any one plane. Thus

$$\rho = x\alpha + y\beta + z\gamma \quad . \quad . \quad . \quad . \quad . \quad (164)$$

refers to any point in space.

If the variable scalars x, y, z are functions of *two* independent variable scalars, as t and u, ρ is the vector to a surface, which, if the functions are linear, will be a plane. We may, therefore, write

$$\rho = \phi(t, u) \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

as the general equation of a surface.

If x, y and z are functions of *one* independent variable scalar, as t, ρ is the vector to a curve, which, if the functions are linear, becomes a right line. We may, therefore, write

$$\rho = \phi(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

as the general equation of a curve in space.

If α, β, γ are complanar, we may replace either two of the vectors in Equation (164) by a single vector, in which case $\rho = \phi(t)$ contains but two variable scalars, functions of t, and is the equation of a plane curve, or of a straight line if the functions are linear.

The essential characteristic of the various equations of a straight line is that they are *linear*, and involve, explicitly or implicitly, *one* indeterminate scalar.

61. Assuming

$$\rho = x\alpha + y\beta, \qquad (a)$$

in which x and y are variable scalars, functions of a single variable and independent scalar, as t, as the general form of the equation of a plane curve, by substituting in any particular case the known functions $x = f(t)$, $y = f'(t)$, or $x = f''(y)$, we may avail ourselves of the Cartesian forms and apply to the resulting function in ρ the reasoning of the Quaternion method.

For example, suppose α and β are unit vectors along the axis and directrix of a parabola, the origin being taken at the focus. In this case we have the Cartesian relation

$$y^2 = 2px + p^2, \qquad (b)$$

or, substituting in (a),

$$\rho = \frac{1}{2p}(y^2 - p^2)\,\alpha + y\beta,$$

as the vector equation of the parabola.

Or, again, α and β being any given vectors parallel to a diameter and tangent at its vertex,

$$\rho = \frac{t^2}{2}\alpha + t\beta \qquad (c)$$

is the vector equation of a parabola, in terms of a single independent scalar t.

62. Let $f(x)$ be any scalar function as, for example,

$$f(x) = x^2.$$

Then

$$d[f(x)] = 2x\,dx = [f'(x)]\,dx.$$

If, however, $f(q)$ be a function of a quaternion q, as, for example, in the above case,

$$f(q) = q^2,$$

then

$$f(q + dq) = (q + dq)^2 = q^2 + q\,dq + dq \cdot q + (dq)^2,$$
$$\therefore d[f(q)] = q\,dq + dq \cdot q,$$

which cannot, however, be written $2qdq$, because of the non-commutative character of quaternion multiplication. We cannot, therefore, write, in general,

$$d[f(q)] = [f'(q)]\,dq,$$

or form, as usual, a differential coefficient. Since vector, as well as quaternion, multiplication is non-commutative, the same is true of the differentiation of a function of a vector. Thus, if

$$f(\rho) = \rho^2,$$
$$d[f(\rho)] = \rho d\rho + d\rho \,.\, \rho,$$

and in order to write $d[f(\rho)] = [f'(\rho)]d\rho$, it would be necessary to determine a vector σ, such that $\sigma d\rho = d\rho \,.\, \rho$, or

$$\sigma = d\rho \,.\, \rho d\rho^{-1},$$

or, if ϵ be the versor of $d\rho$, since the tensors cancel,

$$\sigma = \epsilon\rho\epsilon^{-1};$$

that is (Art. 56, 18), we must have ρ, ϵ and σ complanar, or $V\epsilon\sigma = V\rho\epsilon$. Since complanar quaternions are commutative, if q and dq are complanar, or if dq or $d\rho$ is a scalar, this peculiarity of quaternion and vector differentiation disappears. In this case, dq and $d\rho$ being scalars, $f(q)$ or $f(\rho)$ are quaternion or vector functions of scalar variables, to which the ordinary rules of differentiation are applicable. In fact we have only to assume such a function, as

$$\rho = x'\alpha' + x''\alpha'' + x'''\alpha''' + \cdots\cdots = \Sigma x\alpha = \phi(t),$$

in which α', α'', α''', ... are constants and the only variables are the scalar multipliers, to see that the vectors α', α'', α''' ..... are to be treated as constants and the usual rules of differentiation applied to the scalar coefficients.

Such equations, then, as those of the parabola, (b) and (c), Art. 61, in which α and β are given constant vectors, may be differentiated as usual. Thus, from

$$\rho = \frac{t^2}{2}\alpha + t\beta,$$

we have

$$\frac{d\rho}{dt} = t\alpha + \beta.$$

ρ and ρ' being any two vectors to the curve,

$$\rho' - \rho = \Delta\rho$$

is the vector secant; so that when ρ and ρ' become consecutive, and the secant a tangent,

$$d\rho = (t\alpha + \beta)dt$$

is a vector along the tangent to the curve at the point corresponding to t. The vector to this point being $\frac{t^2}{2}\alpha + t\beta$, and x any variable scalar, we may write the equation of the tangent line at that point

$$\rho = \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta);$$

for any given point, x being the only scalar variable.

63. It has been seen that the usual definition of differential coefficients is inapplicable to quaternions in general, for this definition involves the commutative property of multiplication, which is not, in general, true of quaternions, nor of the vectors to which they may degrade. It becomes necessary, therefore, to give a definition of differentials which shall not involve this property, yet which shall also be true of quaternions which degrade to scalars, and therefore be equally applicable to ordinary scalar quantities.

If $p = f(q)$, such a definition is involved in the formula

$$dp = \underset{n=\infty}{\text{limit}}\, n\left[f(q + n^{-1}dq) - f(q)\right] \quad . \quad . \quad (167),$$

for, let $f(q, r, s, \cdots\cdots) = 0$ be any relation between a system of quaternions $q, r, s, \cdots\cdots$, and let $\Delta q, \Delta r, \Delta s, \cdots\cdots$ be finite and simultaneous differences, so that $q + \Delta q$, $r + \Delta r$, $s + \Delta s$, ... satisfy the relation $f(q, r, s, \cdots\cdots) = 0$. Then in passing from the new system $q + \Delta q$, ... to the old system q, ..., the simultaneous differences can all be made to approach zero together, since they all vanish together. If, while these differences Δq, Δr, ... thus decrease indefinitely together, they be all multiplied by the same increasing number, n, the equimultiples $n\Delta q$, $n\Delta r$, ... may tend to finite limits, and these limits are defined to be the simultaneous differentials of the related quaternions q, r, s, ..., and are written dq, dr, ds, ... Simultaneous differentials are, therefore, the limits of equimultiples of simultaneous decreasing differences. If, then, in $\Delta p = f(q + \Delta q) - f(q)$, while the finite differences Δp, Δq be indefinitely decreased, they be multiplied by a number, n, ultimately to be made infinity, so that

$$n\Delta p = n\,[f(q + \Delta q) - f(q)],$$

and we pass to the limit, writing dp for $n\Delta p$, and dq for $n\Delta q$, we have

$$dp = \underset{n=\infty}{\text{limit}}\, n\left[f\left(q + \frac{dq}{n}\right) - f(q)\right],$$

a formula for the differential of a single explicit function of a single variable.

If $Q = F(q, r, \ldots\ldots)$,

$$dQ = \underset{n=\infty}{\text{limit}}\, n[F(q+n^{-1}dq,\ r+n^{-1}dr, \ldots\ldots) - F(q, r, \ldots\ldots)] \quad (168).$$

In these formulae, dq, dr, ... are any assumed variables, no reference having been made to their magnitudes, and n any positive whole number conceived so as to tend to infinity. To show that these differentials need not be small, as also the application of the formula to the differentiation of ordinary scalar quantities, let

$$y = x^2;$$

then

$$(y+\triangle y)=(x+\triangle x)^2;$$

whence, as usual,

$$\triangle y=2x\triangle x+(\triangle x)^2,$$

or, n being a positive whole number,

$$n\triangle y=2xn\triangle x+n^{-1}(n\triangle x)^2.$$

If, now, the differences $\triangle y$ and $\triangle x$ tend together to zero, while n increases and tends to infinity in such a manner that $n\triangle x$ tends to some finite limit, as a, we have, for the other equimultiple $n\triangle y$,

$$n\triangle y=2xa+n^{-1}a^2.$$

But, since a, and therefore a^2, is finite, $n^{-1}a^2$ tends to zero, and, at the limit, $n\triangle y=2xa$. Hence the limits of the equimultiples $n\triangle x$ and $n\triangle y$ are respectively a and $2xa$, and $dx=a$, $dy=2xa$ by definition; from which

$$dy=2xdx.$$

For a vector function we should write

$$d\rho=\underset{n=\infty}{\text{limit}}\, n\left[f(\rho+n^{-1}d\rho)-f(\rho)\right] \quad . \quad . \quad (169),$$

and for a scalar function, $\rho=\phi(t)$,

$$d\rho=d\left[\phi(t)\right]=\underset{n=\infty}{\text{limit}}\, n\left[\phi\left(t+\frac{dt}{n}\right)-\phi(t)\right] \quad . \quad (170),$$

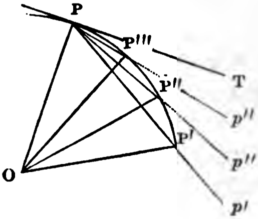
in which letter t and dt are independent and arbitrary scalars.

64. As a further illustration of the definition, let

$$\rho=\phi(t)$$

be the equation of any plane curve in space, and OP $= \rho$ (Fig. 64) a vector from the origin to a point P of the curve; t being any arbitrary scalar representing time, for example; so that its value, for any other point P$'$ of the curve, represents the interval elapsed from any definite epoch to the time when the point generating the curve has reached P$'$.

Fig. 64.



If ρ' be the vector to P$'$, then

$$\rho' - \rho = \mathrm{PP}' = \Delta\rho$$

is strictly the finite difference between ρ and ρ', and, if the corresponding change in t be Δt,

$$\mathrm{PP}' = (\rho + \Delta\,\rho) - \rho = \Delta\,\rho = \phi(t + \Delta\,t) - \phi(t) = \Delta\,\phi(t)\,;$$

where $\mathrm{OP}' = \phi(t + \Delta\,t)$, and $\Delta\,t$ is the interval from P to P$'$.

In $\tfrac{1}{2}\Delta\,t$, P would have reached some point as P$''$, for which $\mathrm{OP}'' = \phi(t + \tfrac{1}{2}\Delta\,t)$, on the supposition that PP$''$ is described in $\tfrac{1}{2}\Delta\,t$. On the basis of this closer approximation to the velocity at P, P would have been found at p'', had this velocity remained unchanged, such that

$$\mathrm{P}p'' = 2\,\mathrm{PP}'' = 2(\mathrm{OP}'' - \mathrm{OP}) = 2[\phi(t + \tfrac{1}{2}\Delta\,t) - \phi(t)].$$

For a closer approximation to the vector described in $\Delta\,t$ with the velocity at P, suppose at the end of $\tfrac{1}{3}\Delta\,t$ the point is at P$'''$, for which $\mathrm{OP}''' = \phi(t + \tfrac{1}{3}\Delta\,t)$. Under this supposition, the vector described in $\Delta\,t$ would have been

$$\mathrm{P}p''' = 3\,\mathrm{PP}''' = 3(\mathrm{OP}''' - \mathrm{OP}) = 3[\phi(t + \tfrac{1}{3}\Delta\,t) - \phi(t)],$$

and, at the limit, representing the multiple of the diminishing chord by $d\rho$,

$$d\rho = \underset{n=\infty}{\text{limit}}\, n\left[\phi\left(t + \frac{dt}{n}\right) - \phi(t)\right].$$

65. Resuming Equation (167),

$$dp = df(q) = \lim_{n=\infty} n\,[f(q + n^{-1}dq) - f(q)], \qquad (a),$$

the second member may be written $f(q,\ dq)$, but not, as ordinarily, $f(q)dq$.

In $f(q,\ dq)$, dq may be composed of parts, as $q', q'', q''', \dots\dots$, with reference to which $f(q,\ dq) = f(q,\ q' + q'' + \dots\dots)$ is distributive. To prove this, let

$$dq = q' + q'';$$

we are to prove that

$$f(q,\ q' + q'') = f(q,\ q') + f(q,\ q'').$$

Since before passing to the limit, the second member of (a) is a function of n, q and dq, we may express this function by the symbol $f_n(q,\ dq)$, and write

$$f(q,\ dq) = n[f(q + n^{-1}dq) - f(q)] = f_n(q,\ dq),$$

or

$$f(q + n^{-1}dq) = f(q) + n^{-1}f_n(q,\ dq).$$

Replacing dq by q' and q'' in succession, we have

$$f(q + n^{-1}q') = f(q) + n^{-1}f_n(q,\ q'),$$
$$f(q + n^{-1}q'') = f(q) + n^{-1}f_n(q,\ q''),$$

and, following the same law of derivation,

$$f(q + n^{-1}q'' + n^{-1}q') = f(q + n^{-1}q'') + n^{-1}f_n(q + n^{-1}q'',\ q'),$$
$$f(q + n^{-1}q' + n^{-1}q'') = f(q) + n^{-1}f_n(q,\ q' + q''),$$

from which

$$f_n(q,\ q' + q'') = f_n(q,\ q'') + f_n(q + n^{-1}q'',\ q'),$$

the limiting form of which, for $n = \infty$, is

$$f(q,\ q' + q'') = f(q,\ q'') + f(q,\ q') \quad . \quad . \quad (171),$$

which may, in like manner, be extended to the case of

$$dq = q' + q'' + q''' + \ldots\ldots.$$

It follows from the above that, if $p = f(q, xdq)$,

$$f(q, xdq) = xf(q, dq) \quad . \quad . \quad . \quad . \quad (172).$$

If $Q = F(q, r, \ldots\ldots)$, whence, Equation (168),

$$\begin{aligned} dQ &= d\,[F(q, r, \ldots\ldots)] \\ &= \underset{n=\infty}{\text{limit}}\, n\,[F(q + n^{-1}dq, r + n^{-1}dr, \ldots\ldots) - F(q, r, \ldots\ldots)], \end{aligned}$$

the last member will be a linear and homogeneous function of dq, dr, ..., and distributive with reference to each of them. Hence, to differentiate such a function, we do so with reference to each factor, and take the sum of the results obtained, as usual; taking care, however, not to make use of the commutative property. Thus $d(qr) = dq \,.\, r + qdr$, but not $rdq + qdr$.

66. When q is a function of any variable scalar t, representing time, for example, then, if t be given a finite increment $\Delta\, t$, for which the corresponding one of q is $\Delta\, q$, we have

$$\Delta\, q = \Delta\, w + \Delta\, xi + \Delta\, yj + \Delta\, zk;$$

and, if the several parts of the quaternion vary continuously with the independent variable t, at the limit we may form, as usual, the differential coefficient

$$\frac{dq}{dt} = \frac{dw}{dt} + \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k.$$

The successive differential coefficients, as also the partial ones, when $q = \phi(t, v, \ldots\ldots)$, are derived from the quadrinomial form in the same manner.

67. Examples.

1. To find $d\mathbf{T}q$.

$$\frac{d\mathbf{T}q}{dt}=\frac{d\sqrt{w^2+x^2+y^2+z^2}}{dt}$$

$$=\frac{1}{\mathbf{T}q}\left(w\frac{dw}{dt}+x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\frac{1}{\mathbf{T}q}\mathbf{S}\,.\,\frac{dq}{dt}\mathbf{K}q=\mathbf{S}\,.\,\frac{dq}{dt}\;\frac{\mathbf{UK}q\mathbf{T}q}{\mathbf{T}q}$$

$$=\mathbf{T}q\mathbf{S}\,.\,\frac{dq}{dt}\;\frac{1}{\mathbf{U}q\mathbf{T}q}=\mathbf{T}q\mathbf{S}\frac{\frac{dq}{dt}}{q},$$

or

$$\frac{d\mathbf{T}q}{dt}=\mathbf{S}\frac{\frac{dq}{dt}}{\mathbf{U}q}.$$

2. $(\mathbf{T}\rho)^2=-\rho^2$.

The first member being a scalar, we have

$$2\,\mathbf{T}\rho d\mathbf{T}\rho.$$

From the second member

$$d(\rho^2)=\underset{n=\infty}{\text{limit}}\,n\left[(\rho+n^{-1}d\rho)^2-\rho^2\right]$$

$$=\text{limit}\,\rho d\rho+d\rho\,.\,\rho+n^{-1}(d\rho)^2$$

$$=\rho d\rho+d\rho\,.\,\rho=2\,\mathbf{S}\rho d\rho.$$

Equating

$$\mathbf{T}\rho d\mathbf{T}\rho=-\,\mathbf{S}\rho d\rho.$$

From this we may obtain

$$d\mathbf{T}\rho=-\,\mathbf{S}\,.\,\mathbf{U}\rho d\rho=\mathbf{S}\frac{d\rho}{\mathbf{U}\rho},$$

or

$$\frac{d\mathbf{T}\rho}{\mathbf{T}\rho}=\mathbf{S}\frac{d\rho}{\rho}.$$

3. To find $d\mathbf{U}q$. We have

$$\mathbf{T}q\mathbf{U}q=q\,;$$

$$d\mathbf{T}q\,.\,\mathbf{U}q+d\mathbf{U}q\,.\,\mathbf{T}q=dq,$$

whence

$$\frac{d\mathrm{T}q \cdot \mathrm{U}q}{\mathrm{T}q\mathrm{U}q} + \frac{d\mathrm{U}q \cdot \mathrm{T}q}{\mathrm{T}q\mathrm{U}q} = \frac{dq}{q},$$

or

$$\frac{d\mathrm{U}q}{\mathrm{U}q} = \frac{dq}{q} - \frac{d\mathrm{T}q}{\mathrm{T}q},$$

and, substituting from Ex. 2,

$$\frac{d\mathrm{U}q}{\mathrm{U}q} = \frac{dq}{q} - \mathrm{S}\frac{dq}{q};$$

$$\therefore \frac{d\mathrm{U}q}{\mathrm{U}q} = \mathrm{V}\frac{dq}{q},$$

or

$$d\mathrm{U}q = \mathrm{V}\frac{dq}{q} \cdot \mathrm{U}q.$$

4. From the above expressions for $d\mathrm{T}q$ and $d\mathrm{U}q$, we have

$$\begin{aligned} dq &= d\mathrm{T}q \cdot \mathrm{U}q + \mathrm{T}q d\mathrm{U}q \\ &= \left(\mathrm{S}\frac{dq}{\mathrm{U}q} + \mathrm{V}\frac{dq}{\mathrm{U}q}\right)\mathrm{U}q \\ &= \left(\mathrm{S}\frac{dq}{q} + \mathrm{V}\frac{dq}{q}\right)q \end{aligned}$$

as the form under which the differential of a quaternion may always be written.

5. To find $d\mathrm{U}\rho$. We have, from $\rho = \mathrm{T}\rho\mathrm{U}\rho$,

$$\begin{aligned} d\rho &= d\mathrm{T}\rho \cdot \mathrm{U}\rho + \mathrm{T}\rho d\mathrm{U}\rho, \\ \frac{d\rho}{\rho} &= \frac{d\mathrm{T}\rho}{\mathrm{T}\rho} + \frac{d\mathrm{U}\rho}{\mathrm{U}\rho} \\ &= \mathrm{S}\frac{d\rho}{\rho} + \frac{d\mathrm{U}\rho}{\mathrm{U}\rho}, \qquad \text{from Ex. 2,} \end{aligned}$$

or

$$\frac{d\mathrm{U}\rho}{\mathrm{U}\rho} = \frac{d\rho}{\rho} - \mathrm{S}\frac{d\rho}{\rho} = \mathrm{V}\frac{d\rho}{\rho} = \mathrm{V}\frac{\rho d\rho}{\rho^2} = \frac{\mathrm{V}\rho d\rho}{\rho^2} = \text{etc.},$$

whence, also,

$$d\mathrm{U}\rho = \frac{\rho\mathrm{V} \cdot d\rho\rho}{(\mathrm{T}\rho)^3}.$$

6. From the above expressions for $d\mathbf{T}\rho$ and $d\mathbf{U}\rho$,

$$d\rho = d\mathbf{T}\rho \,.\, \mathbf{U}\rho + \rho\frac{\mathbf{V} \,.\, d\rho\rho}{(\mathbf{T}\rho)^2}.$$

7. That $\mathbf{S}$, $\mathbf{V}$ and $\mathbf{K}$ are commutative with d is seen from the following:

$$q = \mathbf{S}q + \mathbf{V}q,$$

whence

$$dq = d\mathbf{S}q + d\mathbf{V}q, \qquad (a)$$

and, since dq is a quaternion,

$$dq = \mathbf{S}dq + \mathbf{V}dq, \qquad (b)$$

hence

$$d\mathbf{S}q = \mathbf{S}dq \quad \text{and} \quad d\mathbf{V}q = \mathbf{V}dq. \qquad (c)$$

Again

$$\mathbf{K}q = \mathbf{S}q - \mathbf{V}q,$$

whence

$$d\mathbf{K}q = d\mathbf{S}q - d\mathbf{V}q,$$

and, taking the conjugate of dq in either (b) or (a), we have, with or without (c),

$$d\mathbf{K}q = \mathbf{K}dq.$$

8. $(\mathbf{T}q)^2 = q\mathbf{K}q.$

$$\begin{aligned} 2\,\mathbf{T}q d\mathbf{T}q &= \underset{n=\infty}{\text{limit}}\; n\left[(q + n^{-1}dq)(\mathbf{K}q + n^{-1}d\mathbf{K}q) - q\mathbf{K}q\right] \\ &= \text{limit}\left[dq(\mathbf{K}q + n^{-1}\mathbf{K}dq) + q\mathbf{K}dq\right] \\ &= dq \,.\, \mathbf{K}q + q\mathbf{K}dq \\ &= \mathbf{K} \,.\, q\mathbf{K}dq + q\mathbf{K}dq \\ &= 2\,\mathbf{S} \,.\, q\mathbf{K}dq = 2\,\mathbf{S} \,.\, \mathbf{K}qdq, \qquad \text{[Equation (80)]} \end{aligned}$$

or, since $\mathbf{T}q = \mathbf{TK}q$ and $\mathbf{UK}q = \mathbf{U}\frac{1}{q} = \frac{1}{\mathbf{U}q}$,

$$d\mathbf{T}q = \mathbf{S} \,.\, \mathbf{U}\frac{1}{q}dq = \mathbf{S} \,.\, \mathbf{U}q^{-1}dq.$$

If $q =$ a vector, as ρ, then, since $\mathbf{K}\rho = -\rho$, this becomes

$$d\mathbf{T}\rho = -\mathbf{S} \,.\, \mathbf{U}\rho d\rho,$$

as in Ex. 2.

9. $r = q^2$.

$$\begin{aligned} dr &= \underset{n=\infty}{\text{limit}}\, n\left[(q+n^{-1}dq)^2 - q^2\right] \\ &= \text{limit}\left[qdq + dq \,.\, q + n^{-1}(dq)^2\right] \\ &= qdq + dq \,.\, q; \\ \therefore\ dr &= 2\,\mathbf{S}qdq + 2\,\mathbf{S}q\mathbf{V}dq + 2\,\mathbf{S}dq\mathbf{V}q. \end{aligned}$$

If $q =$ a vector, as ρ, then $\mathbf{S}q = 0$, $\mathbf{S}dq = 0$, and

$$d(\rho^2) = 2\,\mathbf{S}\rho d\rho$$

as in Ex. 2.

10. $r = \sqrt{q}$. Then $q = r^2$, and, as before,

$$dq = rdr + dr \,.\, r.$$

Operating with $r\times$ and $\times\mathbf{K}r$, in succession,

$$\begin{aligned} rdq &= r^2dr + rdr \,.\, r, \\ dq \,.\, \mathbf{K}r &= rdr \,.\, \mathbf{K}r + dr \,.\, r\mathbf{K}r \\ &= rdr \,.\, \mathbf{K}r + (\mathbf{T}r)^2dr, \end{aligned}$$

or, adding,

$$\begin{aligned} rdq + dq \,.\, \mathbf{K}r &= [r^2 + (\mathbf{T}r)^2]dr + rdr(r + \mathbf{K}r) \\ &= [r^2 + (\mathbf{T}r)^2 + 2\,\mathbf{S}r \,.\, r]dr, \end{aligned}$$

which gives $dr = d\sqrt{q}$ in terms of dq.

11. $qq^{-1} = 1$. We have

$$qd(q^{-1}) + dq \,.\, q^{-1} = 0.$$

Operating with $q^{-1}\times$

$$\begin{aligned} q^{-1}qd(q^{-1}) + q^{-1}dq \,.\, q^{-1} &= 0, \\ d\frac{1}{q} &= -\frac{1}{q}dq \,.\, \frac{1}{q}. \end{aligned}$$

If $q =$ a vector, as ρ,

$$\begin{aligned}
d\frac{1}{\rho} &= -\frac{1}{\rho}d\rho\frac{1}{\rho} \\
&= -\frac{1}{\rho}d\rho\frac{1}{\rho} + \frac{1}{\rho^2}d\rho - \frac{1}{\rho}\ \frac{1}{\rho}d\rho \\
&= \frac{d\rho}{\rho^2} - \frac{1}{\rho}\left(\frac{1}{\rho}d\rho + d\rho \,.\, \frac{1}{\rho}\right) \\
&= \frac{d\rho}{\rho^2} - \frac{2}{\rho}\mathbf{S}\frac{d\rho}{\rho} \\
&= \frac{1}{\rho}\left(\frac{d\rho}{\rho} - 2\,\mathbf{S}\frac{d\rho}{\rho}\right) = -\frac{1}{\rho}\mathbf{K}\frac{d\rho}{\rho}.
\end{aligned}$$

12. Differentiate $\mathbf{SU}q$.

$$\begin{aligned}
d\mathbf{SU}q = \mathbf{S}d\mathbf{U}q &= \mathbf{S} \,.\, \mathbf{V}\frac{dq}{q}\mathbf{U}q \qquad \text{[Exs. 7 and 3.]} \\
&= \mathbf{S} \,.\, \frac{dq}{q}\mathbf{VU}q \\
&= \mathbf{S} \,.\, \frac{dq}{q}\mathbf{UV}q\mathbf{TVU}q \\
&= -\mathbf{S} \,.\, \frac{dq}{q\mathbf{UV}q}\mathbf{TVU}q.
\end{aligned}$$

13. Differentiate $\mathbf{VU}q$.

$$\begin{aligned}
d\mathbf{VU}q = \mathbf{V} \,.\, d\mathbf{U}q &= \mathbf{V} \,.\, \mathbf{V}\frac{dq}{q}\mathbf{U}q \qquad \text{[Exs. 7 and 3.]} \\
&= \mathbf{V} \,.\, \mathbf{U}q^{-1}\mathbf{V}(dq \,.\, q^{-1}).
\end{aligned}$$

14. Differentiate $\mathbf{TVU}q$.

$$\begin{aligned}
d\mathbf{TVU}q &= \mathbf{S}\frac{d(\mathbf{VU}q)}{\mathbf{UV}q} \qquad \text{[Ex. 2.]} \\
&= \mathbf{S}\frac{d\mathbf{U}q}{\mathbf{UV}q} = \mathbf{S}\frac{\mathbf{V}\frac{dq}{q}\mathbf{U}q}{\mathbf{UV}q} \\
&= \mathbf{S} \,.\, \frac{dq}{q\mathbf{UV}q}\mathbf{SU}q.
\end{aligned}$$

The Right Line.

As in Cartesian coördinates, the form of the equations of a right line, as of other loci, will depend upon the assumed constants, and in any given problem one form may be more conveniently used than another.

68. Right line through the origin.

If O be the initial point, or origin, and $\rho = \text{OR}$ a variable vector in the prolongation of $\alpha = \text{OA}$, then

$$\rho = x\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

is the equation of a right line through the origin in the direction of the constant vector α.

The equations

$$\left.\begin{aligned} \mathrm{U}\rho &= \mathrm{U}\alpha \\ \mathrm{V}\alpha\rho &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

obviously refer to the same right line.

Since any line, represented as a vector by α, is parallel to $\rho = x\alpha$, we may say that the above equations are those of a right line through the origin parallel to a given line; or, A being a point given by $\alpha = \text{OA}$, they are the equations of a right line through the origin and a given point.

69. Parallel lines.

If $\beta = \text{OB}$ be a constant vector to a given point B, then

$$\rho = \beta + x\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

is the equation of a right line through a given point, and parallel to a given line, as $\rho' = x\alpha$ through the origin. Or, α being a given vector, it is the equation of a right line through a given point and having a given direction. If α is an undetermined vector, it becomes the general equation of any one of the infinite number of right lines which may be drawn through a given point. If O and B coincide, $\beta = 0$, and, as before, $\rho = x\alpha$.

α remaining the same, and $\beta' = $ OB$'$ being a vector to any other point B$'$, for the equations of two parallels, we have

$$\left.\begin{aligned}\rho &= \beta + x\alpha \\ \rho &= \beta' + x'\alpha\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176),$$

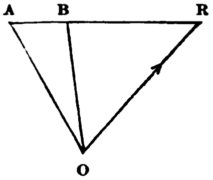
or, since α and $\rho - \beta$ are parallel,

$$\left.\begin{aligned}V\alpha(\rho - \beta) &= 0 \\ V\alpha(\rho - \beta') &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177).$$

70. Right line through two given points.

If OA $= \alpha$ (Fig. 65), OB $= \beta$ are the vectors to the given points, and ρ the variable vector to any point R of the line whose equation is required, we have

Fig. 65.



$$\text{AR} = x\text{AB} = x(\beta - \alpha),$$

and

$$\text{OR} = \text{OA} + \text{AR},$$

or, for the required equation,

$$\rho = \alpha + x(\beta - \alpha) \quad . \quad (178),$$

which, if one of the points, as A, coincides with the origin, becomes $\rho = x\beta$, as before.

We have seen, Art. 55, that if $S\alpha\beta\gamma = 0$, α, β and γ are complanar. Replacing γ by the variable vector ρ,

$$S\alpha\beta\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

is *the equation of a plane*, since it expresses the condition that ρ is complanar with α and β. If we have also $S\alpha\gamma\rho = 0$, the two equations, taken together, represent the line of intersection of these two planes.

These equations may be obtained from the line $\rho = x\alpha$ by operating with $S(V\alpha\beta)\times$ and $S(V\alpha\gamma)\times$; or, conversely, to find the equation of the line in terms of known quantities, having given

$$S\alpha\beta\rho = 0, \quad S\alpha\gamma\rho = 0,$$

write these latter under the form

$$\mathbf{S} \,.\, \rho \mathbf{V} \alpha\beta = 0, \quad \mathbf{S} \,.\, \rho \mathbf{V} \alpha\gamma = 0,$$

whence it appears that ρ is perpendicular to both $\mathbf{V}\alpha\beta$ and $\mathbf{V}\alpha\gamma$, and is consequently parallel to the axis of their product; therefore

$$\begin{aligned} \rho &= y\mathbf{V} \,.\, \mathbf{V}\alpha\beta\mathbf{V}\alpha\gamma \\ &= y(\gamma \mathbf{S}\alpha\beta\alpha - \alpha\mathbf{S}\alpha\beta\gamma) \qquad \text{[Eq. (112)]} \\ &= -y\alpha\mathbf{S}\alpha\beta\gamma, \end{aligned}$$

or, putting $-y\mathbf{S}\alpha\beta\gamma = x$,

$$\rho = x\alpha.$$

71. Right line perpendicular to a given line.

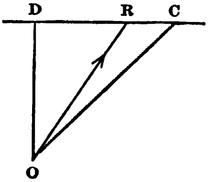
1. Let $\delta = \text{OD}$ (Fig. 66) be a vector through the origin. To find the equation of DC through its extremity and perpendicular to it. Now $\rho - \delta$ is a vector along DR, and therefore by condition

$$\mathbf{S}\delta(\rho - \delta) = 0.$$

Whence $\mathbf{S}\delta\rho = -(\mathbf{T}\delta)^2$, or

$$\mathbf{S}\delta\rho = c, \textit{ a constant} \quad . \quad . \quad . \quad . \quad . \quad (180).$$

Fig. 66.



In order that ρ, $\rho - \delta$ and δ be complanar, we must have

$$\mathbf{S} \,.\, \delta\rho(\rho - \delta) = 0,$$

or

$$\mathbf{S} \,.\, (\mathbf{V}\delta\rho)(\rho - \delta) = 0.$$

2. $\rho - \delta$, being perpendicular to both δ and $\mathbf{V}\delta\rho$, will be parallel to the axis of their product, or to $\mathbf{V} \,.\, \delta\mathbf{V}\delta\rho$. Hence, if $\gamma = \text{OC}$ be a vector to any point C, in the plane of OD and DR, the equation of a right line through a given point C, perpendicular to a given line OD, will be

$$\rho = \gamma + x\mathbf{V} \,.\, \delta\mathbf{V}\delta\gamma \quad . \quad . \quad . \quad . \quad . \quad (181).$$

3. If the perpendicular is to pass through the origin, then, from Equation (180),

$$\mathbf{S}\delta\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (182),$$

or, in another form, from Equation (181), γ being parallel to $\mathbf{V} \,.\, \delta\mathbf{V}\delta\gamma$,

$$\rho = y\mathbf{V} \,.\, \delta\mathbf{V}\delta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (183).$$

4. The student will find it useful to translate the Quaternion into the Cartesian forms. Thus, from Equation (180), if ROD $= \theta$,

$$\mathbf{S}\delta\rho = -\,\mathbf{T}\delta\mathbf{T}\rho \cos\theta,$$

whence, if r and d represent the tensors,

$$rd \cos\theta = d^2, \quad \text{or} \quad r = \frac{d}{\cos\theta},$$

the polar equation of a right line.

5. Equation (181), of a line through a given point and perpendicular to a given line through the origin, may be otherwise obtained, as follows:

Let γ and δ, as before, be vectors to the point and along the given line, respectively, and β a vector along the required perpendicular, whose equation will then be

$$\rho = \gamma + x\beta. \qquad (a)$$

To eliminate β we have the conditions

$$\mathbf{S}\delta\beta = 0,$$

since δ and β are perpendicular to each other, and

$$\mathbf{S}\gamma\delta\beta = 0,$$

since γ, δ and β are complanar. But $\mathbf{V}\delta\gamma$ is perpendicular to this plane, and therefore $\mathbf{V} \,.\, \delta\mathbf{V}\delta\gamma$ is parallel to β; hence, substituting in (a),

$$\rho = \gamma + x\mathbf{V} \,.\, \delta\mathbf{V}\delta\gamma,$$

or simply

$$\rho = \gamma + x\delta\mathbf{V}\delta\gamma.$$

If $S\gamma\delta\beta \lesseqgtr 0$, γ, δ and β are not complanar, and the problem is indeterminate; which also appears from (a), by operating with $\times S \,.\, \delta$, whence, since $S\beta\delta = 0$,

$$S\rho\delta = S\gamma\delta,$$

a result which is independent of β, and an infinite number of lines satisfy the condition.

6. If the line to which the perpendicular is drawn does not pass through the origin, let

$$\rho = \beta + x\alpha \qquad (a)$$

be its equation. Then, if ρ be the vector to the foot of the perpendicular, we have $S\alpha(\rho - \gamma) = 0$, or

$$S\alpha(x\alpha + \beta - \gamma) = 0, \qquad (b)$$

because the line is perpendicular to (a), or its parallel α. Hence, from (b),

$$x\alpha = \alpha^{-1}S\alpha(\gamma - \beta),$$

or, for the perpendicular $\rho - \gamma$,

$$\begin{aligned}\rho - \gamma = x\alpha + \beta - \gamma &= \alpha^{-1}S\alpha(\gamma - \beta) - \alpha^{-1}\alpha(\gamma - \beta)\\ &= -\alpha^{-1}V\alpha(\gamma - \beta).\end{aligned}$$

Its length is evidently

$$TV[U\alpha \,.\, (\gamma - \beta)] \quad . \quad . \quad . \quad . \quad . \quad (184).$$

7. This perpendicular is the shortest distance from the point to the line. The problem may, therefore, be stated thus: to find the shortest distance from C to the line $\rho = x\alpha + \beta$. ρ being the vector to the foot of a line from C to any point of the given line, this vector is

$$\beta + x\alpha - \gamma,$$

and, in order that its length be a minimum,

$$\begin{aligned} dT(\beta + x\alpha - \gamma) &= 0\\ = T(\beta + x\alpha - \gamma)\, dT(\beta + x\alpha - \gamma)&\\ = -S[(\beta + x\alpha - \gamma)\alpha]dx &= 0,\end{aligned}$$

or

$$S(\beta + x\alpha - \gamma)\alpha = 0,$$

that is, the line must be perpendicular to $\rho = x\alpha + \beta$.

8. If the perpendicular distance from the origin to $\rho = \beta + x\alpha$ is required, ρ, being as before the vector to the foot of the perpendicular, coincides with it; hence, γ being zero, and δ representing this value of ρ,

$$\delta = x\alpha + \beta.$$

Operating with $\times\, S\,.\,\delta$, since $S\alpha\delta = 0$,

$$-(T\delta)^2 = S\beta\delta.$$

Hence

$$T\delta = \frac{S\beta\delta}{T\delta} = \frac{S\,.\,\beta T\delta U\delta}{T\delta},$$

or

$$T\delta = S\,.\,\beta U\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185).$$

72. We are to observe that the foregoing equations of a right line are, as remarked in Art. 60, all linear functions involving, explicitly or implicitly, a single real and independent variable scalar. Such is evidently the case for such equations as

$$\rho = x\alpha, \qquad [\text{Eq. } (173)]$$
$$\rho = \beta + x\alpha, \qquad [\text{Eq. } (175)]$$
$$\rho = \alpha + x(\beta - \alpha). \qquad [\text{Eq. } (178)]$$

So also for the implicit forms, as $V\alpha\rho = 0$ [Eq. (174)]; employing the trinomial forms

$$\alpha = ai + bj + ck,$$
$$\rho = xi + yj + zk,$$

we have

$$\alpha\rho = (bz - cy)i + (cx - az)j + (ay - bx)k - (ax + by + cz).$$

Whence

$$V\alpha\rho = (bz - cy)i + (cx - az)j + (ay - bx)k = 0;$$
$$\therefore\; bz = cy, \quad cx = az, \quad ay = bx,$$

in which x and y are functions of z.

The Plane.

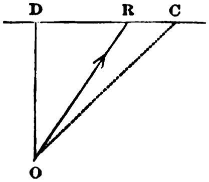
73. *Equation of a plane.*

1. If, in the equation $\mathbf{S}\,.\,\delta\beta = 0$, which denotes that β is perpendicular to δ, we replace β by the variable vector ρ,

$$\mathbf{S}\,.\,\delta\rho = 0\,.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\quad (186)$$

is the equation of a plane through the origin perpendicular to δ.

Fig. 66 (*bis*).



2. Or, let $\delta = $ OD (Fig. 66) be the vector-perpendicular on the plane, and DR any line of the plane.

Then

$$\mathbf{S}\delta(\rho - \delta) = 0,$$
$$\mathbf{S}\delta\rho = \delta^2 = -(\mathbf{T}\delta)^2,$$

or

$$\mathbf{S}\delta\rho = c, \textit{ a constant}\quad .\;\;.\quad (187)$$

is the general equation of a plane perpendicular to δ. Here DR is any line of the plane; and, if $\mathbf{V}\delta\rho = \epsilon$,

$$\mathbf{S}\epsilon\rho = \textit{an indeterminate quantity}\;\;.\;\;.\;\;.\quad (188).$$

If the plane pass through the origin, we have, as before, $\mathbf{S}\delta\rho = 0$. Conversely, if $\mathbf{S}\delta\rho = c$ is the equation of a plane, δ is a vector perpendicular to the plane.

3. The equation of a plane through the origin perpendicular to δ may also be written in terms of any two of its vectors, as γ and β;

$$\rho = x\beta + y\gamma.$$

Both of these indeterminate vectors may be eliminated by operating with $\mathbf{S}\,.\,\delta\,\times$, whence

$$\mathbf{S}\delta\rho = 0$$

as before; or one may be eliminated by operating with $\mathbf{V}\,.\,\beta\,\times$, whence

$$\mathbf{V}\beta\rho = z\delta,$$

from which we may again derive $S\delta\rho = 0$ by operating with $V \, . \, \delta \times$; for

$$\begin{aligned} V \, . \, \delta V\beta\rho = Vz\delta^2 &= 0 \\ &= \rho S\delta\beta - \beta S\delta\rho, \qquad \text{[Eq. (111)]} \end{aligned}$$

whence, since $S\delta\beta = 0$, $S\delta\rho = 0$.

4. The equation of a plane through a point B, for which OB $= \beta$, and perpendicular to δ, is

$$S\delta(\rho - \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (189).$$

5. Having the equation of a plane, $S\delta\rho = c$, to find its distance from the origin, or the length of ρ when it coincides with δ, we have $\rho = x\delta$; hence

$$S\delta\rho = c = Sx\delta^2 = x\delta^2,$$

or

$$x = \frac{c}{\delta^2},$$

which, in $\rho = x\delta$, gives

$$\rho = \frac{c}{\delta},$$

or

$$T\rho = \frac{c}{T\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (190).$$

74. *To find the equation of a plane through the origin, making equal angles with three given lines.*

Let α, β, γ be unit vectors along the lines. The equation of the plane will be of the form

$$S\delta\rho = 0.$$

By condition, $S\alpha\delta = S\beta\delta = S\gamma\delta = T\delta \sin\phi = x$, ϕ being the common angle made by the lines with the plane.

Hence

$$\sin\phi = \frac{x}{T\delta}.$$

To eliminate δ, we have, from Equation (118),

$$\delta S\alpha\beta\gamma = V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta,$$

and, by condition,

$$\delta S\alpha\beta\gamma = x(V\alpha\beta + V\beta\gamma + V\gamma\alpha).$$

The vector represented by the parenthesis is, then, the perpendicular on the plane, whose equation, therefore, is

$$S\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) = 0 \quad . \quad . \quad . \quad . \quad (191),$$

and the sine of the angle ϕ is

$$\frac{S\alpha\beta\gamma}{T(V\alpha\beta + V\beta\gamma + V\gamma\alpha)}.$$

75. *Equation of a plane through three given points.*

Let α, β, γ be vectors to the given points; then are the lines joining these points, as $(\alpha - \beta)$, $(\beta - \gamma)$, lines of the plane. If ρ is the variable vector to any point of the plane, $\rho - \alpha$ is also a line of the plane. Hence

$$S(\rho - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$

or

$$S(\rho\alpha\beta - \rho\alpha\gamma - \rho\beta^2 + \rho\beta\gamma - \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 - \alpha\beta\gamma) = 0.$$

But

$$S(-\rho\beta^2) = 0, \; S(-\alpha^2\beta) = 0, \text{ etc.},$$
$$S(-\rho\alpha\gamma) = S\rho\gamma\alpha = S \,.\, \rho V\gamma\alpha,$$
$$S\rho\alpha\beta = S \,.\, \rho V\alpha\beta, \text{ etc.},$$

hence

$$S \,.\, \rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) - S\alpha\beta\gamma = 0 \quad . \quad . \quad (192),$$

which, by making the vector-parenthesis $= \delta$, may be written under the form

$$S\rho\delta - S\alpha\beta\gamma = 0,$$

in which δ is along the perpendicular from the origin on the plane. When ρ coincides with this perpendicular, $\rho = x\delta$, and, from the above equation,

$$x\delta^2 = S\alpha\beta\gamma,$$

or, for the vector-perpendicular,

$$\rho = x\delta = \delta^{-1} S\alpha\beta\gamma = \frac{S\alpha\beta\gamma}{V\alpha\beta + V\beta\gamma + V\gamma\alpha}.$$

76. We observe again, from inspection of the equations of a plane, that, as remarked in Art. 60, they are linear and functions of two indeterminate scalars. Thus, for a plane through the origin

$$S\delta\rho = 0, \qquad [\text{Eq. } (186)]$$

employing the trinomial forms $\delta = ai + bj + ck$ and $\rho = xi + yj + zk$, we obtain

$$\delta\rho = (bz - cy)i + (cx - az)j + (ay - bx)k - (ax + by + cz),$$

the last term of which is the scalar part; hence

$$ax + by + cz = 0,$$

the equation of a plane through the origin O, perpendicular to a line from O to (a, b, c), which may be written $f(x, y, z) = 0$; or as a function of two indeterminates. In the same way, from an inspection of the other forms,

$$\rho = x\alpha + y\beta, \qquad [\text{Art. } 73, 3]$$
$$\rho = \delta + x\alpha + y\beta,$$
$$S\delta\rho - c' = ax + by + cz - c' = 0, \qquad [\text{Eq. } (187)]$$

we observe they are linear functions of two indeterminate scalars.

77. Exercises and Problems on the Right Line and Plane.

1. *β and γ being vectors along two given lines which intersect at the point* A, *to which the vector is* $OA = \alpha$, *to write the equation of a line perpendicular to each of the two given lines at their intersection.*

$V\beta\gamma$ is a vector in the direction of the required line, whose equation, therefore, is

$$\rho = \alpha + xV\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (193).$$

If $\alpha' = \text{OA}'$ be a vector to any other point, as A,' then is

$$\rho = \alpha' + xV\beta\gamma$$

the equation of a line through a given point perpendicular to a given plane; the latter being given by two of its lines.

2. *α and β being vectors to two given points,* A *and* B, *and* $S\delta\rho = c$ *the equation of a given plane, to find the equation of a plane through* A *and* B *perpendicular to the given plane.*

δ, $\rho - \alpha$ and $\alpha - \beta$ are lines of the required plane, hence

$$S(\rho - \alpha)(\alpha - \beta)\delta = 0,$$

or

$$S\rho(\alpha - \beta)\delta + S\alpha\beta\delta = 0 \quad . \quad . \quad . \quad . \quad . \quad (194)$$

is the required equation.

3. OC $= \gamma$ *being a vector to a given point* C, *and* $\rho = \alpha + x\beta$, $\rho = \alpha' + x'\beta'$ *the equations of two given lines, to write the equation of a plane through* C *parallel to the two given lines.*

If lines be drawn through the given point parallel to the given lines, they will lie in the required plane. As vectors, β and β' are such lines, and $\rho - \gamma$ is also a line of the plane. Hence

$$S\beta\beta'(\rho - \gamma) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

is the required equation. If $\gamma = \alpha$, or α', it is the equation of a plane through one line parallel to the other. Or, if γ is indeterminate, the general equation of a plane parallel to two given lines.

Otherwise: the equation of a plane through the extremity of γ parallel to two given lines, whose directions are given by α and β, is evidently $\rho = \gamma + x\alpha + y\beta$.

4. *To find the distance between two points.*

α and β being vectors to the points,

$$\gamma = \beta - \alpha.$$

Squaring

$$c^2 = b^2 + a^2 - 2ab \cos \text{C}.$$

5. *A plane being given by two of its lines, β and γ, to write the equation of a right line through* A *perpendicular to the plane.*

Let OA $= \alpha$. Draw two lines through A parallel to β and γ. Then

$$\rho = \alpha + x\mathbf{V}\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (196).$$

If the plane is given by the equation $\mathbf{S}\delta\rho = c$, then

$$\rho = \alpha + x\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (197).$$

6. *Find the length of the perpendicular from* A *to the plane, in the preceding example.*

Operating on Equation (197) with $\mathbf{S}\,.\,\delta\,\times$

$$\mathbf{S}\delta\rho = \mathbf{S}\delta\alpha + x\delta^2 = c,$$

or

$$x\delta^2 = c - \mathbf{S}\delta\alpha\,;$$
$$\therefore\ x\delta = \delta^{-1}(c - \mathbf{S}\delta\alpha) \quad . \quad . \quad . \quad . \quad . \quad (198).$$

7. $\mathbf{S}\delta(\rho - \beta) = 0$, *Equation* (189), *being the equation of a plane through* B *perpendicular to* δ, *to find the distance from a point* C *to the plane.*

Let $\gamma =$ OC. The perpendicular on the plane from C, being parallel to δ, will have for its equation

$$\rho = \gamma + x\delta.$$

To find x, operate with $\mathbf{S}\,.\,\delta\,\times$, whence

$$\mathbf{S}\delta\rho = \mathbf{S}\delta\gamma + x\delta^2,$$

or, from the equation of the plane,

$$\mathbf{S}\delta\gamma + x\delta^2 = \mathbf{S}\delta\beta\,;$$
$$\therefore\ x\delta = -\,\delta^{-1}\mathbf{S}\delta(\gamma - \beta),$$

and

$$x\mathbf{T}\delta = \mathbf{T}\delta^{-1}\mathbf{S}\delta(\gamma - \beta) = \mathbf{S}\,[\mathbf{U}\delta\,.\,(\gamma - \beta)].$$

8. *Write the equation of a plane through the parallels*

$$\rho = \alpha + x\beta,$$
$$\rho = \alpha' + x\beta.$$

9. *Write the equation of a plane through the line*

$$\rho = \alpha + x\beta$$

perpendicular to the plane

$$S\delta\rho = 0.$$

10. *Given the direction of a vector-perpendicular to a plane, to find its length so that the plane may meet three given planes in a point.*

Let δ be the given vector-perpendicular, and

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c$$

the equations of the given planes. If the equation of the plane be written

$$S\delta\rho = x,$$

then x must have such a value that one value of ρ shall satisfy the equations of all four of the planes. From Equation (118) we have

$$\begin{aligned}\rho S\alpha\beta\gamma &= V\alpha\beta S\gamma\rho + V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho \\ &= cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha.\end{aligned}$$

Operating with $S \,.\, \delta \times$, to introduce x,

$$xS\alpha\beta\gamma = cS\delta\alpha\beta + aS\delta\beta\gamma + bS\delta\gamma\alpha.$$

11. *To find the shortest distance between two given right lines.*

Let the lines be given by the equations

$$\rho = \alpha + x\beta, \qquad (a)$$

$$\rho = \alpha' + x'\beta'. \qquad (b)$$

The equation of a plane through either line, as (b), parallel to the other (a), is [Equation (195)]

$$S\beta\beta'(\rho - \alpha') = 0. \qquad (c)$$

$V\beta\beta'$ is a vector-perpendicular to this plane. Hence, if $yV\beta\beta'$ be the shortest vector distance between the lines, we have, since $\alpha - \alpha' - yV\beta\beta'$ is a vector complanar with β and β',

$$S\beta\beta'(\alpha - \alpha' - yV\beta\beta') = 0,$$

or

$$S(S\beta\beta' + V\beta\beta')(\alpha - \alpha' - yV\beta\beta') = 0,$$

whence

$$-y(V\beta\beta')^2 + S[V\beta\beta'(\alpha - \alpha')] = 0;$$

or, dividing by $T(V\beta\beta')$,

$$T(yV\beta\beta') = TS[(\alpha - \alpha')U(V\beta\beta')] \quad . \quad . \quad . \; (199),$$

the symbol T denoting that only the positive numerical value of the scalar is taken.

Otherwise: since the distance is to be a minimum,

$$dT(\rho' - \rho) = 0,$$

whence

$$S(\rho' - \rho)(\beta' dx' - \beta dx) = 0,$$

or

$$S(\rho' - \rho)\beta = 0 \quad \text{and} \quad S(\rho' - \rho)\beta' = 0,$$

or the shortest distance is their common perpendicular, whose length may be found as above.

12. *Given* $S\delta_1\rho = d_1$ *and* $S\delta_2\rho = d_2$, *the equations of two planes, to find the equation of their line of intersection.*

This equation will be of the form

$$\rho = m\delta_1 + n\delta_2 + xV\delta_1\delta_2. \qquad (a)$$

To find m and n, we have, from (a),

$$S\delta_1\rho = m\delta_1^2 + nS\delta_1\delta_2,$$
$$S\delta_2\rho = n\delta_2^2 + mS\delta_1\delta_2,$$

from which we obtain

$$m = \frac{S\delta_1\rho - nS\delta_1\delta_2}{\delta_1^2} = \frac{S\delta_2\rho - n\delta_2^2}{S\delta_1\delta_2};$$

$$\therefore \; n = \frac{S\delta_1\delta_2 S\delta_1\rho - \delta_1^2 S\delta_2\rho}{(S\delta_1\delta_2)^2 - \delta_1^2\delta_2^2}.$$

But

$$(S\delta_1\delta_2)^2 - \delta_1^2\delta_2^2 = (V\delta_1\delta_2)^2;$$

$$\therefore \; n = \frac{d_1 S\delta_1\delta_2 - d_2\delta_1^2}{(V\delta_1\delta_2)^2}.$$

And similarly

$$m = \frac{d_2 \mathbf{S}\delta_1\delta_2 - d_1\delta_2^2}{(\mathbf{V}\delta_1\delta_2)^2}.$$

Substituting these values in (a)

$$\rho = \frac{d_1 \mathbf{S}\delta_1\delta_2 - d_2\delta_1^2}{(\mathbf{V}\delta_1\delta_2)^2}\delta_2 + \frac{d_2 \mathbf{S}\delta_1\delta_2 - d_1\delta_2^2}{(\mathbf{V}\delta_1\delta_2)^2}\delta_1 + x\mathbf{V}\delta_1\delta_2,$$

which is the equation of the required line, a less useful form than those of the two simple conditions of Art. 70.

If the two planes pass through the origin, then also does their line of intersection; and since every line in one plane is perpendicular to δ_1, and every line in the other to δ_2, $\mathbf{V}\delta_1\delta_2$ is a line along the intersection, as in (a), and the equation becomes

$$\rho = x\mathbf{V}\delta_1\delta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (200).$$

13. *The planes being given as in Equation* (189),

$$\mathbf{S}\delta(\rho - \beta) = 0, \qquad (a)$$
$$\mathbf{S}\delta'(\rho - \beta') = 0, \qquad (b)$$

to find the line of intersection.

The vector ρ to any point of the line must satisfy both (a) and (b). This vector may be decomposed into three vectors parallel to δ, δ' and $\mathbf{V}\delta\delta'$, which are given, and not complanar, by Equation (118); whence

$$\rho\mathbf{S} . \delta\delta'\mathbf{V}\delta\delta' = \mathbf{S}\rho\delta\mathbf{V}(\delta' . \mathbf{V}\delta\delta') + \mathbf{S}\rho\delta'\mathbf{V}(\mathbf{V}\delta\delta' . \delta) + \mathbf{S}(\rho\mathbf{V}\delta\delta')\mathbf{V}\delta\delta',$$

or, from (a) and (b),

$$-\rho(\mathbf{TV}\delta\delta')^2 = \mathbf{S}\delta\beta\mathbf{V}(\delta' . \mathbf{V}\delta\delta') + \mathbf{S}\delta'\beta'\mathbf{V}(\mathbf{V}\delta\delta' . \delta) + \mathbf{S}\delta\delta'\rho\mathbf{V}\delta\delta',$$

or, since $\mathbf{S}\delta\delta'\rho$ is the only indeterminate scalar, putting it equal to x, we have

$$-\rho(\mathbf{TV}\delta\delta')^2 = \mathbf{S}\delta\beta\mathbf{V}(\delta' . \mathbf{V}\delta\delta') + \mathbf{S}\delta'\beta'\mathbf{V}(\mathbf{V}\delta\delta' . \delta) + x\mathbf{V}\delta\delta'.$$

If the planes pass through the origin, in which case β and β' are zero, we have, as before,

$$\rho = x\mathbf{V}\delta\delta'.$$

14. *To write the equation of a plane through the origin and the line of intersection of*

$$S\delta(\rho-\beta)=0, \qquad (a)$$
$$S\delta'(\rho-\beta')=0. \qquad (b)$$

If ρ is such that $S\delta\rho = S\delta\beta$, then also $S\delta'\rho = S\delta'\beta'$, and both the above equations will be satisfied. Hence, from (a) and (b)

$$S\delta\rho S\delta'\beta' - S\delta\beta S\delta'\rho = 0,$$

which is also a plane through the origin. This equation may also be written

$$S\left[(\delta S\delta'\beta' - \delta' S\delta\beta)\rho\right]=0,$$

which shows that

$$\delta S\delta'\beta' - \delta' S\delta\beta$$

is a vector-perpendicular to the plane, and therefore to the line of intersection of (a) and (b).

15. *To find the equation of condition that four points lie in a plane.*

If the vectors to the four points be α, β, γ, δ, then, to meet the condition,

$$\delta-\alpha, \quad \delta-\beta, \quad \delta-\gamma$$

must be complanar, and therefore

$$S(\delta-\alpha)(\delta-\beta)(\delta-\gamma)=0,$$

whence

$$S\delta\beta\gamma + S\alpha\delta\gamma + S\alpha\beta\delta = S\alpha\beta\gamma \quad . \quad . \quad . \quad (201),$$

which is the equation of condition.

Or, x and y being indeterminate, we have also

$$\delta = \alpha + x(\beta-\alpha) + y(\gamma-\beta),$$

or

$$\delta + (x-1)\alpha + (y-x)\beta - y\gamma = 0,$$

and

$$1 + (x-1) + (y-x) - y = 0.$$

Or, in general,

$$\left.\begin{aligned} a\alpha + b\beta + c\gamma + d\delta = 0 \\ a + b + c + d = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (202),$$

are the sufficient conditions of complanarity.

These conditions are analogous to Equation (9).

16. *Given the three planes of a triedral, to find the equations of planes through the edges perpendicular to the opposite faces, and to show that they intersect in a right line.*

Taking the vertex as the initial point, let

$$\mathbf{S}\alpha\rho = 0, \qquad (a)$$
$$\mathbf{S}\beta\rho = 0, \qquad (b)$$
$$\mathbf{S}\gamma\rho = 0 \qquad (c)$$

be the equations of the plane faces. Then $\mathbf{V}\alpha\beta$ is a vector parallel to the intersection of (a) and (b), and $\mathbf{V} \,.\, \gamma\mathbf{V}\alpha\beta$ is a vector perpendicular to the required plane through their common edge. Hence the equation of this plane is

$$\mathbf{S}(\rho\mathbf{V} \,.\, \gamma\mathbf{V}\alpha\beta) = 0. \qquad (a')$$

Similarly, or by a cyclic change of vectors,

$$\mathbf{S}(\rho\mathbf{V} \,.\, \alpha\mathbf{V}\beta\gamma) = 0, \qquad (b')$$
$$\mathbf{S}(\rho\mathbf{V} \,.\, \beta\mathbf{V}\gamma\alpha) = 0 \qquad (c')$$

are the equations of the other two planes.

If from their common point of intersection normals are drawn to the planes, then are $\mathbf{V} \,.\, \gamma\mathbf{V}\alpha\beta$, $\mathbf{V} \,.\, \alpha\mathbf{V}\beta\gamma$ and $\mathbf{V} \,.\, \beta\mathbf{V}\gamma\alpha$ vector lines parallel to them; but, Equation (123),

$$\mathbf{V}(\gamma\mathbf{V}\alpha\beta + \alpha\mathbf{V}\beta\gamma + \beta\mathbf{V}\gamma\alpha) = 0.$$

Hence these vectors are complanar, and the planes therefore intersect in a right line.

Otherwise: from Equation (111)

$$\mathbf{V}(\alpha\mathbf{V}\beta\gamma) = \gamma\mathbf{S}\alpha\beta - \beta\mathbf{S}\alpha\gamma;$$

hence, from (b'),

$$S(\rho\gamma S\alpha\beta - \rho\beta S\alpha\gamma) = S\alpha\beta S\rho\gamma - S\alpha\gamma S\rho\beta = 0.$$

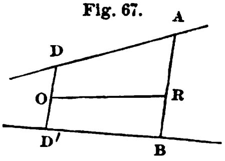
Similarly, or by cyclic permutation,

$$S\beta\gamma S\rho\alpha - S\beta\alpha S\rho\gamma = 0,$$
$$S\gamma\alpha S\rho\beta - S\gamma\beta S\rho\alpha = 0.$$

But the sum of these three equations is identically zero, either two giving the third by subtraction or addition.

17. *To find the locus of a point which divides all right lines terminating in two given lines into segments which have a common ratio.*

Fig. 67.



Let DA and D′B (Fig. 67) be the two given lines, α and β unit vectors parallel to them, BA any line terminating in the given lines, and R a point such that RA = mBR. Assume DD′, a perpendicular to both the given lines, O, its middle point, as the origin, and OD $= \delta$, OD′ $= -\delta$, OR $= \rho$.

Then

$$\text{OA} = \rho + \text{RA} = \delta + x\alpha.$$
$$\text{OB} = \rho + \text{RB} = -\delta + y\beta.$$

Adding

$$2\rho + \text{RA} + \text{RB} = x\alpha + y\beta. \qquad (a)$$

But

$$\text{RA} + \text{RB} = \frac{m-1}{m}\text{RA} = \frac{m-1}{m}(-\rho + \delta + x\alpha),$$

which substituted in (a) gives

$$\rho - \delta - x\alpha = m(y\beta - \rho - \delta),$$

whence, since $S\delta\beta = S\delta\alpha = 0$,

$$S\delta\rho(m+1) = \delta^2(1-m) = c,$$

or *the locus is a plane perpendicular to* DD′.

If the given ratio is unity, or BR = RA, then $m = 1$ and

$$S\delta\rho = 0,$$

and the locus is a plane through O perpendicular to DD'.

If α and β are parallel, then (b) becomes

$$\rho - \delta = m(x'\alpha - \rho - \delta),$$

whence

$$S\delta\rho(m + 1) = (1 - m)\delta^2,$$

a right line perpendicular to DD'. If at the same time $m = 1$,

$$S\delta\rho = 0 \quad \text{and} \quad \rho = x''\alpha,$$

a right line through the origin parallel to the given lines.

18. *If the sums of the perpendiculars from two given points on two given planes are equal, the sum of the perpendiculars from any point of the line joining them is the same.*

Let A and B be the given points, OA $= \alpha$, OB $= \beta$, and $S\delta\rho = d$, $S\delta'\rho = d'$ be the equations of the planes; δ and δ' being unit vectors, so that $x\delta$ and $y\delta'$ are the vector-perpendiculars from A on the planes. Then

$$x = S\alpha\delta - d,$$
$$y = S\alpha\delta' - d';$$

and

$$x + y = S\alpha(\delta + \delta') - (d + d').$$

Similarly

$$x' + y' = S\beta(\delta + \delta') - (d + d').$$

But, by condition,

$$S\alpha(\delta + \delta') = S\beta(\delta + \delta'),$$

or

$$S(\beta - \alpha)(\delta + \delta') = 0. \qquad (a)$$

The vector from O to any other point of the line AB is $\alpha + z(\beta - \alpha)$; whence, for this point,

$$x'' + y'' = S[\alpha + z(\beta - \alpha)](\delta + \delta') - (d + d'),$$

for which point, since (a) remains true, the sum therefore is unchanged.

19. *To find the locus of the middle points of the elements of an hyperbolic paraboloid.*

Let the equations of the plane director and rectilinear directrices be

$$S\delta\rho = 0,$$

$$\rho = \alpha + x\beta \quad \text{and} \quad \rho = \alpha' + x'\beta'.$$

Also, let OM $= \mu$ be the vector to the middle point of an element so chosen that the vectors to the extremities are $\alpha + x\beta$ and $\alpha' + x'\beta'$. Then, since M is the middle point,

$$2\mu = \alpha + x\beta + \alpha' + x'\beta'. \qquad (a)$$

The vector element is

$$-x'\beta' - \alpha' + \alpha + x\beta,$$

and, being parallel to the plane director,

$$S\delta(-\alpha' + \alpha + x\beta - x'\beta') = 0.$$

This is a scalar equation between known quantities from which we may find x' in terms of x; substituting this value in (a), we have an equation of the form

$$\mu = \alpha_1 + x\beta_1,$$

or the locus is a right line.

20. *If, from any three points on the line of intersection of two planes, lines be drawn, one in each plane, the triangles formed by their intersections are sections of the same pyramid.*

The Circle and Sphere.

78. *Equations of the circle.*

The equation of the circle may be written under various forms. If α and β are vector-radii at right angles to each other, and $T\alpha = T\beta$, we may write

$$\rho = \cos\theta \, . \, \alpha + \sin\theta \, . \, \beta \quad . \quad . \quad . \quad . \quad (203),$$

in terms of a single variable scalar θ.

If α and β are unit vectors along the radii,

$$\rho = x\alpha + y\beta,$$

or, since $x^2 + y^2 = r^2$,

$$\rho = (r^2 - y^2)^{\frac{1}{2}}\alpha + y\beta \quad . \quad . \quad . \quad . \quad (204).$$

The initial point being at the center,

$$\left.\begin{array}{l} \mathrm{T}\rho = \mathrm{T}\alpha \\ \mathrm{T}\dfrac{\rho}{\alpha} = 1 \\ \rho^2 = -r^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

are evidently all equations of the circle.

If O (Fig. 68) be any initial point, C the center, to which the vector OC $= \gamma$, ρ the variable vector to any point P, CP $= \alpha$, then

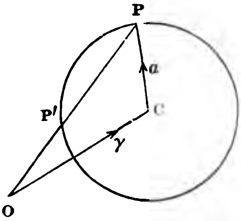
$$\rho - \gamma = \alpha,$$

whence

$$(\rho - \gamma)^2 = -r^2 \quad . \quad . \quad (206),$$

the vector equation of the circle whose radius is r.

Fig. 68.



If $\mathrm{T}\gamma = c$, it may be put under the form

$$\rho^2 - 2\,\mathrm{S}\rho\gamma = c^2 - r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (207).$$

If the initial point is on the circumference, we still have $(\rho - \gamma)^2 = -r^2$; but $\gamma^2 = -r^2$, hence

$$\rho^2 - 2\,\mathrm{S}\rho\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208).$$

or, since in this case $\mathrm{S}\rho\gamma = \mathrm{S}\rho\alpha$,

$$\rho^2 - 2\,\mathrm{S}\rho\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209).$$

79. *Equations of the sphere.*

This surface may be conveniently treated of in connection with the circle; for, since nothing in the previous article restricts the lines to one plane, the equations there deduced for the circle are also applicable to the sphere.

Another convenient form of the equation of a sphere is (Fig. 68)

$$T(\rho - \gamma) = T\alpha \quad . \quad . \quad . \quad . \quad . \quad (210),$$

the center being at the extremity of γ and $T\alpha$ the radius.

80. *Tangent line and plane.*

A vector along the tangent being $d\rho$, we have, from Equation (203),

$$d\rho = -\sin\theta \cdot \alpha + \cos\theta \cdot \beta,$$

and for the tangent line $\pi = \rho + xd\rho$,

$$\pi = \cos\theta \cdot \alpha + \sin\theta \cdot \beta + x[-\sin\theta \cdot \alpha + \cos\theta \cdot \beta] \quad (211),$$

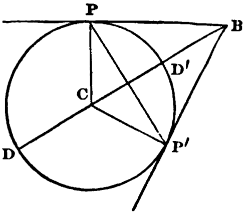
where π is any vector to the tangent line at the point corresponding to θ.

From the above we have directly

$$S\rho d\rho = 0,$$

or the tangent is perpendicular to the radius vector drawn to the point of tangency.

Fig. 69.



By means of this property we may write the equation of the tangent as follows: let π be the vector to any point of the tangent, as B (Fig. 69), C being the initial point and ρ the vector to P, the point of tangency. Then

$$S\rho(\pi - \rho) = 0,$$

whence

$$\left.\begin{aligned} S\rho\pi &= -r^2 \\ S\frac{\pi}{\rho} &= 1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (212),$$

or

are the equations of a tangent. Since nothing restricts the line to one plane, they are also the equations of the tangent plane to a sphere.

The above well-known property may also be obtained by differentiating $T\rho = T\alpha$; whence, Art. 67, 2,

$$S\rho d\rho = 0,$$

and therefore ρ is perpendicular to the tangent line or plane.

81. *Chords of contact.*

In Fig. 69 let CB $= \beta$ be the vector to a given point. The equation of the tangent BP must be satisfied for this point; hence, from Equation 212,

$$S\beta\rho = -r^2,$$

or

$$S\beta\sigma = -r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213),$$

which is equally true of the other point of tangency P,' and being the equation of a right line, it is that of the chord of contact PP.' And for the reason previously given, it is also the equation of the plane of the circle of contact of the tangent cone to the sphere, the vertex of the cone being at B.

82. Exercises and Problems on the Circle and the Sphere.

In the following problems the various equations of the plane, line, circle and sphere are employed to familiarize the student with their use. Other equations than those selected in any special problem might have been used, leading sometimes more directly to the desired result. It will be found a useful exercise to assume forms other than those chosen, as also to transform the equations themselves and interpret the results. Thus, for example, the equation of the circle (209),

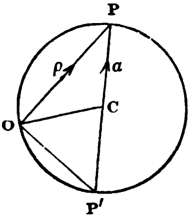
$$\rho^2 - 2\,S\rho\alpha = 0$$

may be transformed into

$$S\rho(\rho - 2\,\alpha) = 0,$$

which gives immediately (Fig. 70) the property of the circle, that the angle inscribed in a semi-circle is a right angle. Obviously, this includes the case of chords drawn from any point in a sphere to the extremities of a diameter, and the above equation is a statement of the proposition that, ρ being a variable vector, the locus of the vertex of a right angle, whose sides pass through the extremities of α and $-\alpha$, is a sphere.

Fig. 70.



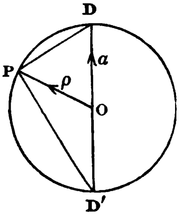
Again, with the origin at the center, we have (Fig. 71),

$$(\rho+\alpha)+(\alpha-\rho)=2\alpha,$$

and, operating with $\times\, \mathbf{S} \,.\, (\rho-\alpha)$,

$$\mathbf{S}(\rho+\alpha)(\rho-\alpha)=0\,;$$

Fig. 71.



$\therefore$ P is a right angle. This also follows from $\mathbf{T}\rho=\mathbf{T}\alpha$, whence $\rho^2=\alpha^2$ and $\mathbf{S}(\rho+\alpha)(\rho-\alpha)=0$.

Again, from $\mathbf{T}\rho=\mathbf{T}\alpha$,

$$\mathbf{T}(\rho+\alpha)(\rho-\alpha)=2\,\mathbf{TV}\alpha\rho.$$

The first member is the rectangle of the chords PD, PD' (Fig. 71), and the second member is

$$2\,\mathrm{OD}\,.\,\mathrm{OP}\sin\mathrm{DOP}.$$

Hence the rectangle on the chords drawn from any point of a circle to the extremities of a diameter is four times the area of a triangle whose sides are ρ and α.

Also, from $\mathbf{T}\rho=\mathbf{T}\alpha$,

$$\rho^2=-r^2,$$

and for any other point

$$\rho'^2=-r^2;$$

$$\therefore\ \mathbf{S}(\rho'+\rho)(\rho'-\rho)=0.$$

But $\rho'-\rho$ is a vector along the secant, and $\rho'+\rho$ is a vector along the angle-bisector; now when the secant becomes a tangent, the angle-bisector becomes the radius; therefore the radius to the point of contact is perpendicular to the tangent.

1. *The angle at the center of a circle is double that at the circumference standing on the same arc.*

We have

$$T\rho = T\alpha,$$

and therefore, Art. 56, 18,

$$\rho = (\rho + \alpha)^{-1}\alpha(\rho + \alpha),$$

whence the proposition.

2. *In any circle, the square of the tangent equals the product of the secant and its external segment.*

In Fig. 69 we have

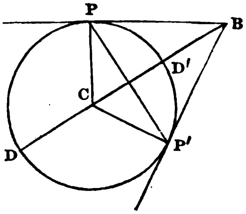
$$\text{CB} = \text{CP} + \text{PB},$$

$$\therefore \text{CB}^2 = \text{CP}^2 + \text{PB}^2,$$

or

$$\begin{aligned}\text{PB}^2 &= \text{CB}^2 - \text{CP}^2\\ &= \text{CB}^2 - \text{CD}^2, \text{ as lines,}\\ &= \text{BD} \cdot \text{BD}'.\end{aligned}$$

Fig. 69 (*bis*).



3. *The right line joining the points of intersection of two circles is perpendicular to the line joining their centers.*

Let (Fig. 72) $\text{CC}' = \alpha$, $\text{CP} = \rho$, $\text{CP}' = \rho'$, and r, r' be the radii of the circles. Then

$$\rho^2 = -r^2,$$

$$(\rho - \alpha)^2 = -r'^2;$$

also

$$(\rho' - \alpha)^2 = -r'^2.$$

Hence

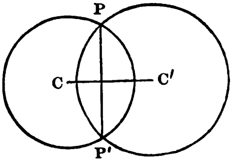
$$S\rho\alpha = S\rho'\alpha,$$

or

$$S\alpha(\rho - \rho') = 0;$$

hence PP' and CC' are at right angles.

Fig. 72.



4. *A chord is drawn parallel to the diameter of a circle; the radii to the extremities of the chord make equal angles with the diameter.*

If ρ and ρ' be the vector-radii, 2α the vector-diameter, then $x\alpha =$ the vector-chord, and

$$(\rho' - x\alpha)^2 = -r^2,$$
$$(\rho + x\alpha)^2 = -r^2,$$

whence the proposition.

5. *If* ABC *is a triangle inscribed in a circle, show that the vector of the product of the three sides in order is parallel to the tangent at the initial point.* [Compare Art. 55.]

If $\text{AB} = \beta$, $\text{CA} = \gamma$, and O is the center of the circle, then

$$\begin{aligned} -\mathbf{V}(\text{AB} \cdot \text{BC} \cdot \text{CA}) &= \mathbf{V} \cdot \beta(\beta + \gamma)\gamma \\ &= \mathbf{V}(\beta^2\gamma + \beta\gamma^2) \\ &= \beta^2\gamma + \gamma^2\beta. \end{aligned}$$

Or, C and B being points of the circumference satisfying $\rho^2 - 2\mathbf{S}\rho\alpha = 0$ [Eq. (209)], substituting and operating with $\mathbf{S} \cdot \alpha \times$

$$\mathbf{S} \cdot \alpha\mathbf{V}(\text{AB} \cdot \text{BC} \cdot \text{CA}) = 2\mathbf{S}\alpha\beta\mathbf{S}\alpha\gamma - 2\mathbf{S}\alpha\beta\mathbf{S}\alpha\gamma = 0.$$

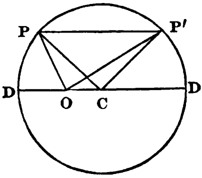
Hence $\mathbf{V}(\text{AB} \cdot \text{BC} \cdot \text{CA})$ is perpendicular to α, or parallel to the tangent at A.

6. *The sum of the squares of the lines from any point on a diameter of a circle to the extremities of a parallel chord is equal to the sum of the squares of the segments of the diameter.*

Let PP′ (Fig. 73) be the chord parallel to the diameter DD′, O the given point, and C the center of the circle. Let $\text{CP} = \rho$, $\text{CP}' = \rho'$, $\text{OC} = \alpha$, $\text{OP} = \beta$ and $\text{OP}' = \beta'$. Then

$$\begin{aligned} \text{OP}^2 &= -\beta^2 = -(\alpha^2 + 2\mathbf{S}\alpha\rho + \rho^2), \\ \text{OP}'^2 &= -\beta'^2 = -(\alpha^2 + 2\mathbf{S}\alpha\rho' + \rho'^2) \,; \\ \therefore\ \text{OP}^2 + \text{OP}'^2 &= 2\,\text{OC}^2 + 2\,\text{DC}^2 - 2(\mathbf{S}\alpha\rho + \mathbf{S}\alpha\rho'). \end{aligned}$$

Fig. 73.



But

$$\mathrm{S}(\rho-\rho')(\rho+\rho')=\mathrm{S}(\rho+\rho')x\alpha=0.$$

Therefore

$$\mathrm{S}\alpha\rho+\mathrm{S}\alpha\rho'=0,$$

and

$$\text{OP}^2+\text{OP}'^2=\text{DO}^2+\text{OD}'^2.$$

7. *To find the intersection of a plane and a sphere.*

Let $\rho^2=-r^2$ be the equation of the sphere, δ a vector-perpendicular from its center on the plane and $\mathrm{T}\delta=d$. Then, if β be a vector of the plane,

$$\rho=\delta+\beta.$$

Substituting in the equation of the sphere, since $\mathrm{S}\beta\delta=0$, we have

$$\beta^2=-(r^2-d^2),$$

the equation of a circle whose radius is $\sqrt{r^2-d^2}$, and which is real so long as $d<r$.

8. *To find the intersection of two spheres.*

Let the equations of the given spheres be (Eq. 207)

$$\rho^2-2\,\mathrm{S}\rho\gamma=c^2-r^2,$$
$$\rho^2-2\,\mathrm{S}\rho\gamma'=c'^2-r'^2.$$

Subtracting, we have

$$2\,\mathrm{S}\rho(\gamma-\gamma')=a\ constant.$$

The intersection is therefore a circle whose plane is perpendicular to $\gamma-\gamma'$, the vector-line joining the centers of the spheres.

Assuming (Eq. 210)

$$\mathrm{T}(\rho-\gamma)=\mathrm{T}\alpha \quad\text{and}\quad \mathrm{T}(\rho-\gamma')=\mathrm{T}\alpha',$$

show that $2\,\mathrm{S}\rho(\gamma-\gamma')=a\ constant$, as above.

9. *The planes of intersection of three spheres intersect in a right line.*

Let γ', γ'', γ''' be the vector-lines to the centers of the spheres, and their equations

$$\begin{aligned} \rho^2 - 2\,\mathrm{S}\rho\gamma' &= c', \\ \rho^2 - 2\,\mathrm{S}\rho\gamma'' &= c'', \\ \rho^2 - 2\,\mathrm{S}\rho\gamma''' &= c'''. \end{aligned}$$

The equations of the planes of intersection are, from the preceding problem,

$$\begin{aligned} 2\,\mathrm{S}\rho(\gamma' - \gamma'') &= c'' - c', & (a) \\ 2\,\mathrm{S}\rho(\gamma' - \gamma''') &= c''' - c', & (b) \\ 2\,\mathrm{S}\rho(\gamma'' - \gamma''') &= c''' - c''. & (c) \end{aligned}$$

Now, if ρ be so taken as to satisfy (a) and (b), we shall obtain their line of intersection. But if ρ satisfies (a) and (b), it will also satisfy their difference, which is (c); the planes therefore intersect in a right line.

10. *To find the locus of the intersections of perpendiculars from a fixed point upon lines through another fixed point.*

Let P and P′ be the points, PP′ $= \alpha$, and δ a vector-perpendicular on any line through P′, as $\rho = \alpha + x\beta$. Then

$$\delta = \alpha + y\beta,$$

and operating with $\mathrm{S}\,.\,\delta\,\times$

$$\delta^2 = \mathrm{S}\delta\alpha,$$

which is the equation of a circle (Eq. 209) whose diameter is PP′.

11. *From a fixed point* P, *lines are drawn to points, as* P′, P″, ... *of a given right line. Required the locus of a point* O *on these lines, such that* PP′ . PO $= m^2$.

Let the variable vector PO$=\rho$; then PP′$=x\rho$. By the condition

$$\mathrm{T}(\text{PP}'\,.\,\text{PO}) = m^2,$$

or

$$\mathrm{T}(x\rho\,.\,\rho) = m^2;$$

$$\therefore\ x\rho^2 = -m^2.$$

If δ be the vector-perpendicular from P on the given line, and $\mathbf{T}\delta = d$,

$$\mathbf{S}\delta(x\rho - \delta) = 0,$$

or

$$x\mathbf{S}\delta\rho = -d^2;$$

$$\therefore \ \rho^2 = \frac{m^2}{d^2}\mathbf{S}\delta\rho;$$

hence the locus is a circle through P.

12. *If through any point chords be drawn to a circle, to find the locus of the intersection of the pairs of tangents drawn at the points of section of the chords and circle.*

Let the point A be given by the vector OA $= \alpha$, O being the initial point taken at the center of the circle. Let $\rho' =$ OR be the vector to one point of intersection R. The locus of R is required. The equation of the chord of contact is (Eq. 213)

$$\mathbf{S}\rho'\sigma = -r^2,$$

which, since the chord passes through A, may be written

$$\mathbf{S}\rho'\alpha = -r^2,$$

where α is a constant vector. The locus is therefore a straight line perpendicular to OA (Eq. 180).

13. *To find the locus of the feet of perpendiculars drawn through a given point to planes passing through another given point.*

Let the initial point be taken at the origin of perpendiculars, α the vector to the point through which the planes are passed, and δ a perpendicular. Then

$$\mathbf{S}\delta(\delta - \alpha) = 0,$$

or

$$\delta^2 - \mathbf{S}\alpha\delta = 0$$

is true for *any* perpendicular. Hence the locus is a sphere whose diameter is the line joining the given points.

Otherwise: if the origin be taken at the point common to the planes, and the equation of one of the planes is $S\delta\rho = 0$, then the vector-perpendicular is (Eq. 198)

$$\delta^{-1}S\delta\alpha,$$

and, if ρ be the vector to its foot,

$$\rho = \alpha - \delta^{-1}S\delta\alpha,$$

or

$$\rho - \alpha = \quad - \delta^{-1}S\delta\alpha,$$

whence

$$(\rho - \alpha)^2 = \quad \delta^{-2}(S\delta\alpha)^2,$$

and

$$S\alpha\rho - \alpha^2 = - \delta^{-2}(S\delta\alpha)^2.$$

Adding the last two equations

$$\rho^2 - S\alpha\rho = 0,$$

or

$$T(\rho - \tfrac{1}{2}\alpha) = T\tfrac{1}{2}\alpha,$$

which is the equation of a sphere whose radius is $T\frac{\alpha}{2}$ and center is at the extremity of $\frac{\alpha}{2}$, or whose diameter is the line joining the points.

14. *To find the locus of a point* P *which divides any line* OS *drawn from a given point to a given plane, so that*

$$\text{OP} \cdot \text{OS} = m, \textit{ a constant.}$$

Let OP $= \rho$ and OS $= \sigma$; also let $S\delta\sigma = c$ be the equation of the plane. We have, by condition,

$$T\rho T\sigma = m,$$

and

$$U\rho = U\sigma;$$

$$\therefore\ T\sigma = \frac{m}{T\rho},$$

and

$$\sigma = \frac{mU\rho}{T\rho}$$

$$= - \frac{m\rho}{\rho^2}.$$

Substituting in the equation of the plane

$$mS\delta\rho + c\rho^2 = 0,$$

which is the equation of a sphere passing through O and having $\frac{m}{\text{OD}}$ for a diameter.

15. *To find the locus of a point the ratio of whose distances from two given points is constant.*

Let O and A be the two given points, OA $= \alpha$, OR $= \rho$, R being a point of the locus. Then, by condition, if m be the given ratio,

$$T(\rho - \alpha) = mT\rho,$$

whence

$$\rho^2 - 2S\rho\alpha + \alpha^2 = m^2\rho^2,$$

$$(1 - m^2)\rho^2 = 2S\alpha\rho - \alpha^2$$

$$= 2S\alpha\rho - \frac{1 - m^2}{1 - m^2}\alpha^2,$$

or

$$\rho^2 - \frac{2S\alpha\rho}{1 - m^2} + \frac{\alpha^2}{(1 - m^2)^2} = \frac{m^2\alpha^2}{(1 - m^2)^2};$$

$$\therefore\ T\left(\rho - \frac{\alpha}{1 - m^2}\right) = T\frac{m\alpha}{1 - m^2},$$

which is the equation of a sphere whose radius is $T\frac{m}{1-m^2}\alpha$, and whose center C is on the line OA, so that OC $= \frac{1}{1-m^2}\alpha$. (Eq. 210).

16. *Given two points* A *and* B, *to find the locus of* P *when*

$$\text{AP}^2 + \text{BP}^2 = \text{OP}^2.$$

O being the origin, let OA $= \alpha$, OB $= \beta$, OP $= \rho$. Then, by condition,

$$\rho^2 = (\rho - \alpha)^2 + (\rho - \beta)^2,$$

whence

$$\rho^2 - 2S\rho(\alpha + \beta) = -(\alpha^2 + \beta^2),$$

$$[\rho - (\alpha + \beta)]^2 = 2S\alpha\beta,$$

$$T[\rho - (\alpha + \beta)] = \sqrt{-2S\alpha\beta},$$

which is the equation of a sphere whose center is at the extremity of $(\alpha+\beta)$, if $\mathbf{S}\alpha\beta$ is negative, or the angle AOB acute. If this angle is obtuse, there is no point satisfying the condition. If AOB = 90°, the locus is a point.

83. *Exercises in the transformation and interpretation of elementary symbolic forms.*

1. From the equation

$$(\rho+\alpha)^2=(\rho-\alpha)^2$$

derive in succession the equations

$$\mathbf{T}(\rho+\alpha)=\mathbf{T}(\rho-\alpha) \quad \text{and} \quad \mathbf{T}\frac{\rho+\alpha}{\rho-\alpha}=1,$$

and state what locus they represent.

2. From the equation

$$\mathbf{K}\frac{\rho}{\alpha}+\frac{\rho}{\alpha}=0$$

derive symbolically the equations

$$\alpha\rho+\rho\alpha=0,\ \mathbf{S}\frac{\rho}{\alpha}=0,\ \mathbf{SU}\frac{\rho}{\alpha}=0,\ \left(\mathbf{U}\frac{\rho}{\alpha}\right)^2=-1, \text{ and } \mathbf{TVU}\frac{\rho}{\alpha}=1,$$

and interpret them as the equations of the same locus.

3. Transform

$$\mathbf{S}\frac{\rho-\alpha}{\alpha}=0$$

to the forms

$$\mathbf{S}\frac{\rho}{\alpha}=1 \quad \text{and} \quad \mathbf{SU}\frac{\rho}{\alpha}=\mathbf{T}\frac{\alpha}{\rho},$$

and interpret.

4. Transform $\mathbf{S}\frac{\rho-\beta}{\alpha}=0$ to $\mathbf{S}\frac{\rho}{\alpha}=\mathbf{S}\frac{\beta}{\alpha}$, and interpret.

5. Transform $(\rho-\beta)^2=(\rho-\alpha)^2$ to $\mathbf{T}(\rho-\beta)=\mathbf{T}(\rho-\alpha)$, and interpret.

6. What locus is represented by $\mathbf{K}\frac{\rho}{\alpha}-\frac{\rho}{\alpha}=0$?

7. What by $\left(\frac{\rho}{\alpha}\right)^2 = -1$? By $\left(\frac{\rho}{\alpha}\right)^2 = -a^2$?

8. What by $\mathbf{U}\frac{\rho}{\alpha} = \mathbf{U}\frac{\beta}{\alpha}$? $\mathbf{U}\rho = \mathbf{U}\beta$? $\mathbf{U}\frac{\rho}{\beta} = 1$?

9. $\mathbf{U}\frac{\rho}{\alpha} = -\mathbf{U}\frac{\beta}{\alpha}$?

10. $\left(\mathbf{U}\frac{\rho}{\alpha}\right)^2 = \mathbf{U}\frac{\gamma}{\alpha}$?

11. $\mathbf{V}\frac{\rho-\beta}{\alpha} = 0$? $\mathbf{V}\frac{\rho}{\alpha} = \mathbf{V}\frac{\beta}{\alpha}$?

12. $\mathbf{V}\frac{\rho}{\alpha} = 0$?

13. $\frac{\rho}{\alpha}\mathbf{K}\frac{\rho}{\alpha} = a^2$?

14. $\mathbf{SU}\frac{\rho}{\alpha} = \mathbf{SU}\frac{\beta}{\alpha}$? $\mathbf{SU}\frac{\rho}{\alpha} = -\mathbf{SU}\frac{\beta}{\alpha}$? $\left(\mathbf{SU}\frac{\rho}{\alpha}\right)^2 = \left(\mathbf{SU}\frac{\beta}{\alpha}\right)^2$?

15. $\mathbf{T}\rho = 1$?

16. Transform $(\rho - \alpha)^2 = \alpha^2$ to $\mathbf{T}(\rho - \alpha) = \mathbf{T}\alpha$, and interpret.

17. Under what other form may we write $(\rho - \alpha)^2 = (\beta - \alpha)^2$? Of what locus is it the equation?

18. $\rho^2 + a^2 = 0$? $\rho^2 + 1 = 0$? Translate the latter into Cartesian coördinates, by means of the trinomial form, and so determine the locus anew.

19. $\mathbf{T}(\rho - \beta) = \mathbf{T}(\beta - \alpha)$?

20. Compare $\mathbf{SU}\frac{\alpha}{\rho} = \mathbf{T}\frac{\rho}{\alpha}$ and $\mathbf{S}\frac{\alpha}{\rho} = 1$ with the forms of Ex. 3.

21. What locus is represented by $\mathbf{S}\beta\rho + \rho^2 = 0$ when $\mathbf{T}\beta = 1$?

22. $\left(\mathbf{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathbf{V}\frac{\rho}{\alpha}\right)^2 = 1$?

23. $\left(\mathbf{T}\frac{\rho}{\alpha}\right)^2 = -1$?

24. Show that $\mathbf{V}\,.\,\mathbf{V}\alpha\beta\mathbf{V}\alpha\rho = 0$ is the equation of a plane. What plane? [Eq. (112)].

The Conic Sections.

Cartesian Forms.

84. *The Parabola.*

Resuming the general form of the equation of a plane curve

$$\rho = x\alpha + y\beta,$$

from the relation $y^2 = 2px$, we obtain

$$\rho = \frac{y^2}{2p}\alpha + y\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

for the vector equation of the parabola when the vertex is the initial point. If the latter is taken anywhere on the curve, from the relation $y^2 = 2p'x$, we obtain

$$\rho = \frac{y^2}{2p'}\alpha + y\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (215);$$

and if the initial point is at the focus, then $y^2 = 2px + p^2$ gives

$$\rho = \frac{1}{2p}(y^2 - p^2)\alpha + y\beta \quad . \quad . \quad . \quad . \quad . \quad (216);$$

or again, in terms of a single scalar t,

$$\rho = \frac{t^2}{2}\alpha + t\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217).$$

In Equations (214), (215) and (216), α and β are *unit* vectors parallel to a diameter and tangent at its vertex, being at right angles to each other in Equations (214) and (216); in Equation (217) α and β are *any* given vectors parallel to a diameter and tangent at its vertex, the initial point being on the curve.

85. *Tangent to the parabola.*

From Equation (216) we have for the vector along the tangent (Art. 62)

$$\frac{y}{p}\alpha + \beta,$$

and, therefore, the equation of the tangent is

$$\pi=\frac{1}{2p}(y^2-p^2)\alpha+y\beta+Y\left(\frac{y}{p}\alpha+\beta\right) \quad . \quad . \quad (218).$$

From Equation (217) the vector along the tangent is

$$t\alpha+\beta,$$

and the equation of the tangent is

$$\pi=\frac{t^2}{2}\alpha+t\beta+x(t\alpha+\beta) \quad . \quad . \quad . \quad . \quad (219).$$

If ρ be the vector to a point on the diameter of a parabola, the point being given by the equation

$$\rho=m\alpha+n\beta, \qquad (a)$$

and a tangent to the curve be drawn through this point, then (a) must satisfy the equation of the tangent-line and

$$m\alpha+n\beta=\frac{t^2}{2}\alpha+t\beta+x(t\alpha+\beta),$$

whence

$$m=\frac{t^2}{2}+xt \quad \text{and} \quad n=t+x,$$

or

$$t=n\pm\sqrt{n^2-2m}; \qquad (b)$$

hence, in general, two tangents can be drawn to the curve through the given point. When $n^2=2m$, they coincide; in this case, from (a),

$$\rho=\frac{n^2}{2}\alpha+n\beta,$$

the point being on the curve. If $2m>n^2$, t is imaginary, and no tangent can be drawn; in this case (a) becomes

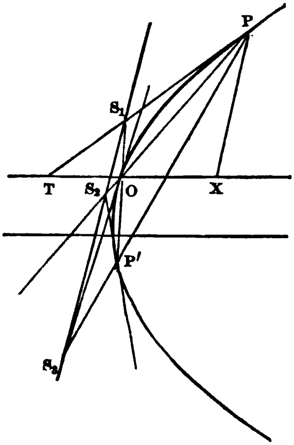
$$\rho=\left(\frac{n^2}{2}+a\right)\alpha+n\beta,$$

the point being within the curve.

86. Examples *on the parabola.*

1. *The intercept of the tangent on the diameter is equal to the abscissa of the point of contact.*

Fig. 74.



Since the tangent is parallel to the vector $t\alpha + \beta$, or to any multiple of it, it is parallel to $t^2\alpha + t\beta$ or to $\frac{t^2}{2}\alpha + t\beta + \frac{t^2}{2}\alpha$, that is, to (Fig. 74)

$$\text{OP} + \text{OX}.$$

But

$$\text{TP} = \text{TO} + \text{OP};$$

$$\therefore\ \text{TO} = \text{OX}.$$

2. *If, from any point on a diameter produced, tangents be drawn, the chord of contact is parallel to the tangent at the vertex of the diameter.*

If t' and t'' correspond to the points of tangency, we have for the vector-chord of contact

$$\rho' - \rho'' = \frac{t'^2}{2}\alpha + t'\beta - \frac{t''^2}{2}\alpha - t''\beta,$$

which is parallel to

$$\beta + \frac{t' + t''}{2}\alpha,$$

or, from Equation (b), Art. 85, to

$$\beta + n\alpha,$$

which is independent of m.

3. *To find the locus of the extremity of the diagonal of a rectangle whose sides are two chords drawn from the vertex.*

Let OP and OP′ be the chords. Then

$$\text{OP} = \rho = \frac{y^2}{2p}\alpha + y\beta, \qquad (a)$$

$$\text{OP}' = \rho' = \frac{y'^2}{2p}\alpha - y'\beta. \qquad (b)$$

The vector-diagonal $\hat{\omega}'$ is $\rho+\rho'$, or

$$\hat{\omega}'=\frac{y^2+y'^2}{2p}\alpha+(y-y')\beta,$$

which may be put under the form of the equation of the parabola by adding and subtracting $\frac{2yy'}{2p}\alpha$, giving

$$\hat{\omega}'=\frac{(y-y')^2}{2p}\alpha+(y-y')\beta+\frac{2yy'}{2p}\alpha. \qquad (c)$$

But, by condition, $S\rho\rho'=0$. Hence, from (a) and (b), $S\alpha\beta$ being zero,

$$yy'-\frac{y^2y'^2}{4p^2}=0, \quad \therefore\ yy'=(2p)^2, \qquad (d)$$

which in (c) gives

$$\hat{\omega}'=\frac{(y-y')^2}{2p}\alpha+(y-y')\beta+4p\alpha.$$

Changing the origin to the extremity of $4p\alpha$,

$$\hat{\omega}=\frac{(y-y')^2}{2p}\alpha+(y-y')\beta.$$

Hence the locus is a similar parabola whose vertex is at a distance of twice the parameter of the given parabola from its vertex.

Moreover, from (d), $xx'=(2p)^2$. Hence *the parameter is a mean proportional between the ordinates and the abscissas of the extremities of chords at right angles.*

4. *If tangents be drawn at the vertices of an inscribed triangle, the sides of the triangle produced will intersect the tangents in three points of a right line.*

Let OPP′ (Fig. 74) be the inscribed triangle, and one of the vertices, as O, the initial point. Then, for the points P and P′ respectively, we have

$$\rho=\frac{t^2}{2}\alpha+t\beta,$$

$$\rho'=\frac{t'^2}{2}\alpha+t'\beta.$$

Let π_1, π_2, π_3 be the vectors to the points of intersection; then

$$\pi_1 = \text{OP} + \text{PS}_1 = \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta).$$

Also

$$\pi_1 = x'\text{OP}' = x'\left(\frac{t'^2}{2}\alpha + t'\beta\right);$$

$$\therefore\ \frac{t^2}{2} + xt = \frac{x't'^2}{2},\quad t + x = x't',$$

$$x' = \frac{t^2}{2tt' - t'^2}.$$

Hence

$$\pi_1 = \frac{t^2}{2tt' - t'^2}\left(\frac{t'^2}{2}\alpha + t'\beta\right) = \frac{t^2}{2t - t'}\left(\frac{t'}{2}\alpha + \beta\right).$$

In a similar manner

$$\pi_2 = \frac{t'^2}{2t' - t}\left(\frac{t}{2}\alpha + \beta\right).$$

But

$$\pi_3 = \text{OP} + y\text{PP}' = \text{OP} + y(\rho' - \rho)$$

$$= \frac{t^2}{2}\alpha + t\beta + y\left[\frac{t'^2 - t^2}{2}\alpha + (t' - t)\beta\right].$$

Also

$$\pi_3 = z\beta;$$

$$\therefore\ \frac{t^2}{2} + y\frac{t'^2 - t^2}{2} = 0,\quad t + y(t' - t) = z,$$

$$z = \frac{tt'}{t + t'}.$$

Hence

$$\pi_3 = \frac{tt'}{t + t'}\beta.$$

Now

$$\frac{2t-t'}{t}\pi_1 - \frac{2t'-t}{t'}\pi_2 - \frac{t^2-t'^2}{tt'}\pi_3 = t\left(\frac{t'}{2}\alpha + \beta\right) - t'\left(\frac{t}{2}\alpha + \beta\right) - (t - t')\beta = 0.$$

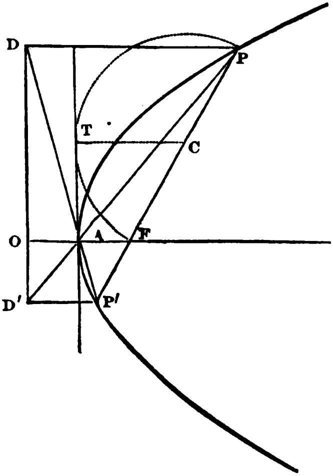
Also

$$\frac{2t - t'}{t} - \frac{2t' - t}{t'} - \frac{t^2 - t'^2}{tt'} = 0.$$

Hence π_1, π_2 and π_3 terminate in a straight line.

5. *The principal tangent is tangent to all circles described on the radii vectores as diameters.*

Fig. 75.



Let AP $= \rho$ (Fig. 75), α and β being unit vectors along the axis and principal tangent. Then, if the circle cut the tangent in T, and TC be drawn to the center,

$$\mathbf{T}(\text{TC}) = \mathbf{T}(\text{FC}) = \mathbf{T}(\tfrac{1}{2}\text{FP});$$
$$\therefore\ \text{TC}^2 = \tfrac{1}{4}(\rho - m\alpha)^2.$$

Also

$$\begin{aligned}\text{TC} &= \text{TA} + \text{AF} + \text{FC}\\ &= -z\beta + m\alpha + \tfrac{1}{2}(\rho - m\alpha),\\ \text{TC}^2 &= [-z\beta + m\alpha + \tfrac{1}{2}(\rho - m\alpha)]^2.\end{aligned}$$

Equating these values of TC², we have, since $S\beta\alpha = 0$,

$$z^2\beta^2 - zS\beta\rho + mS\alpha\rho = 0,$$
$$\therefore\ z^2 - zy + \frac{y^2}{4} = 0,$$

which gives but one value for z.

6. *To find the length of the curve.*

It has been seen (Art. 62) that, if $\rho = \phi(t)$ be the equation of a plane curve, the differential coefficient is the tangent to the curve. Hence, if this be denoted by $\rho' = \phi'(t)$, $\mathbf{T}\rho' dt$ is an element of the curve whose length will be found by integrating $\mathbf{T}\rho'$ with reference to the scalar variable involved between proper limits; or

$$s - s_0 = \int_{t_0}^{t} \mathbf{T}\rho'.$$

For the parabola

$$\rho = \frac{y^2}{2p}\alpha + y\beta,$$

we have

$$\rho' = \frac{y}{p}\alpha + \beta,$$

$$\therefore\ \mathbf{T}\rho' = \sqrt{\frac{y^2}{p^2} + 1} = \frac{1}{p}(y^2 + p^2)^{\frac{1}{2}};$$

$$s - s_0 = \frac{1}{p}\int_{y_0}^{y}(p^2 + y^2)^{\frac{1}{2}} = \left[\frac{y(p^2 + y^2)^{\frac{1}{2}}}{2p} + \frac{p}{2}\log\left(\frac{y + (p^2 + y^2)^{\frac{1}{2}}}{p}\right)\right]_{y_0}^{y}.$$

7. *To find the area of the curve.*

With the notation of the previous example, twice the area swept over by the radius vector will be measured by (Art. 41, 7) $\mathbf{TV}\rho\rho' dt$. The area will then be found by integrating $\mathbf{TV}\rho\rho'$ with reference to the involved scalar between proper limits and taking one-half the result; or

$$A - A_0 = \tfrac{1}{2}\int_{t_0}^{t}\mathbf{TV}\rho\rho'.$$

For the parabola

$$A - A_0 = \tfrac{1}{2}\int_{y_0}^{y}\mathbf{TV}\left(\frac{y^2}{2p}\alpha + y\beta\right)\left(\frac{y}{p}\alpha + \beta\right),$$

or, since $\overset{\wedge}{\alpha\beta} = 90°$,

$$= \tfrac{1}{2}\int_{y_0}^{y}\frac{y^2}{2p} = \frac{1}{12p}\left[y^3\right]_{y_0}^{y}.$$

From the origin, where $y_0 = 0$, to any point whose ordinate is y, the area of the sector swept over by ρ is $\frac{1}{12p}y^3 = \frac{1}{6}xy$; adding the area $\frac{1}{2}xy$ of the triangle, which, with the sector, makes up the total area of the half curve, we have $\frac{2}{3}xy$, or two-thirds that of the circumscribing rectangle. The origin may be changed to any point in the plane of the curve, to which the vector is γ, by substituting the value $\rho = \gamma + \rho_1$ in the equation of the curve, ρ_1 being the new radius vector; we may thus find any sector area limited by two positions of ρ_1, the vertex of the sector being at the new origin. Thus, transferring to an origin on the principal tangent, distant b from the vertex, $\rho = b\beta + \rho_1$; which, in the equation of the parabola, gives

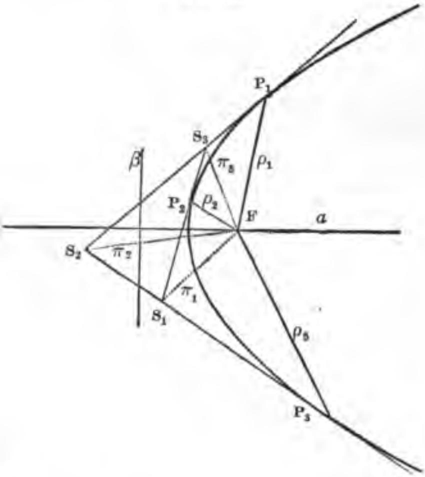
$$\rho_1 = \frac{y^2}{2p}\alpha + (y - b)\beta, \quad \rho_1' = \frac{y}{p}\alpha + \beta;$$

integrating, as before, between the limits $y = b$ and $y = 0$,

$$\tfrac{1}{2}\int_0^y \mathbf{TV}\rho_1\rho_1' = \frac{1}{6p}y^3 = \tfrac{1}{3}xy.$$

87. *Relations between three intersecting tangents to the Parabola.* ["Am. Journal of Math.," vol. i. p. 379. M. L. Holman and E. A. Engler.]

Fig. 76.



Let ρ_1, ρ_2, ρ_3 be the vectors to the three points of tangency, P_1, P_2, P_3 [Fig. 76], and π_1, π_2, π_3 the vectors to S_1, S_2, S_3, the points of intersection of the tangents. Resuming Equation (216), where the focus is the initial point, and a and β are unit vectors along the axis and the directrix,

$$\rho = \frac{1}{2p}(y^2 - p^2)a + y\beta \quad . \quad . \quad . \quad . \quad . \quad (a).$$

Since $\rho^2 = -(\mathbf{T}\rho)^2$, and $\mathbf{S}a\beta = 0$, we have for the three points P_1, P_2, P_3

$$\left.\begin{aligned} \mathbf{T}\rho_1 &= \frac{1}{2p}(y_1^2 + p^2) \\ \mathbf{T}\rho_2 &= \frac{1}{2p}(y_2^2 + p^2) \\ \mathbf{T}\rho_3 &= \frac{1}{2p}(y_3^2 + p^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (b).$$

The vector along the tangent is

$$\frac{y}{p}a + \beta,$$

and therefore

$$\pi_1 = \rho_2 + \mathrm{P}_2\mathrm{S}_1 = \frac{1}{2p}(y_2^2 - p^2)\alpha + y_2\beta + z\left(\frac{y_2}{p}\alpha + \beta\right),$$

$$\pi_1 = \rho_3 + \mathrm{P}_3\mathrm{S}_1 = \frac{1}{2p}(y_3^2 - p^2)\alpha + y_3\beta + w\left(\frac{y_3}{p}\alpha + \beta\right);$$

whence, equating the coefficients of α and β,

$$z = \tfrac{1}{2}(y_3 - y_2), \quad w = \tfrac{1}{2}(y_2 - y_3),$$

whence, substituting, and by the cyclic permutation of the subscripts,

$$\left.\begin{aligned} \pi_1 &= \frac{1}{2p}(y_3 y_2 - p^2)\alpha + \tfrac{1}{2}(y_2 + y_3)\beta \\ \pi_2 &= \frac{1}{2p}(y_1 y_3 - p^2)\alpha + \tfrac{1}{2}(y_3 + y_1)\beta \\ \pi_3 &= \frac{1}{2p}(y_2 y_1 - p^2)\alpha + \tfrac{1}{2}(y_1 + y_2)\beta \end{aligned}\right\} \quad . \ . \ . \ (c).$$

From (b)

$$\left.\begin{aligned} \mathrm{T}\rho_1\mathrm{T}\rho_2 &= \frac{1}{4p^2}(y_1^2 + p^2)(y_2^2 + p^2) \\ \mathrm{T}\rho_2\mathrm{T}\rho_3 &= \frac{1}{4p^2}(y_2^2 + p^2)(y_3^2 + p^2) \\ \mathrm{T}\rho_3\mathrm{T}\rho_1 &= \frac{1}{4p^2}(y_3^2 + p^2)(y_1^2 + p^2) \end{aligned}\right\} \quad . \ . \ . \ (d),$$

and from (c)

$$\left.\begin{aligned} (\mathrm{T}\pi_1)^2 &= \frac{1}{4p^2}(y_2^2 + p^2)(y_3^2 + p^2) \\ (\mathrm{T}\pi_2)^2 &= \frac{1}{4p^2}(y_3^2 + p^2)(y_1^2 + p^2) \\ (\mathrm{T}\pi_3)^2 &= \frac{1}{4p^2}(y_1^2 + p^2)(y_2^2 + p^2) \end{aligned}\right\} \quad . \ . \ . \ (e),$$

and from (d) and (e)

$$\left.\begin{aligned} (\mathrm{T}\pi_3)^2 &= \mathrm{T}\rho_1\mathrm{T}\rho_2 \\ (\mathrm{T}\pi_2)^2 &= \mathrm{T}\rho_1\mathrm{T}\rho_3 \\ (\mathrm{T}\pi_1)^2 &= \mathrm{T}\rho_2\mathrm{T}\rho_3 \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ (f).$$

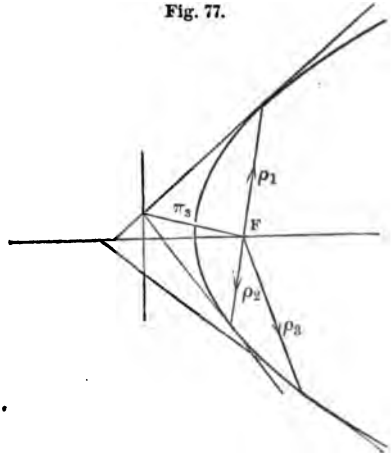
From (c), it appears that *the distance of the point of intersection of two tangents from the axis is the arithmetical mean of the ordinates to their points of contact.* From (f), that *the distance from the focus to the point of intersection of two tangents is a mean proportional to the radii vectores to the points of contact.*

1st. If ρ_2 becomes a multiple of β,

$$\rho_2 = \frac{1}{2p}(y_2^2 - p^2)\alpha + y_2\beta = z\beta;$$
$$\therefore\ z = y_2 = \pm p.$$

Or, *the parameter is the double ordinate tnrough the focus*, or *twice the distance from the focus to the directrix.*

Fig. 77.



2d. If ρ_1 is the multiple of ρ_2 (Fig. 77), then $\rho_2 - \rho_1$ is a focal chord, and

$$x\rho_2 = \rho_1,$$

or, from (a),

$$x\left[\frac{1}{2p}(y_2^2 - p^2)\alpha + y_2\beta\right] = \frac{1}{2p}(y_1^2 - p^2)\alpha + y_1\beta;$$

whence

$$x = \frac{y_1^2 - p^2}{y_2^2 - p^2} = \frac{y_1}{y_2},$$

or

$$y_1(y_1y_2+p^2)=y_2(y_1y_2+p^2),$$

and

$$y_1y_2+p^2=0. \tag{g}$$

From (a) and (c)

$$\begin{aligned} S\pi_3\rho_1 &= -\frac{1}{2p}(y_2y_1-p^2)\frac{1}{2p}(y_1^2-p^2)-\tfrac{1}{2}(y_1+y_2)y_1 \\ &= -\frac{1}{4p^2}(y_1^2+p^2)(y_1y_2+p^2)=0\,; \end{aligned} \tag{h}$$

or, *a line from the focus to the intersection of the tangents at the extremities of a focal chord is perpendicular to the focal chord.*

The vectors along the tangents are

$$\rho_1-\pi_3 \quad \text{and} \quad \rho_2-\pi_3,$$

and, from (h),

$$S(\rho_1-\pi_3)(\rho_2-\pi_3)=S\rho_1\rho_2+\pi_3^2=0,$$

or, *the tangents at the extremities of the focal chord are perpendicular to each other.*

Since, from (g),

$$y_1y_2=-p^2,$$

we have

$$\begin{aligned} \pi_3 &= \frac{1}{2p}(y_1y_2-p^2)\alpha+\tfrac{1}{2}(y_1+y_2)\beta \\ &= -p\alpha+\tfrac{1}{2}(y_1+y_2)\beta, \end{aligned}$$

or, *the tangents at the extremities of a focal chord intersect on the directrix.*

3d. If ρ_2 becomes a multiple of α (Fig. 78), $y_2=0$, and from (c)

$$\begin{aligned} \pi_3 &= \frac{1}{2p}(y_2y_1-p^2)\alpha+\tfrac{1}{2}(y_1+y_2)\beta \\ &= -\frac{p}{2}\alpha+\frac{y_1}{2}\beta, \end{aligned} \tag{i}$$

or, *the subtangent is bisected at the vertex.*

Also

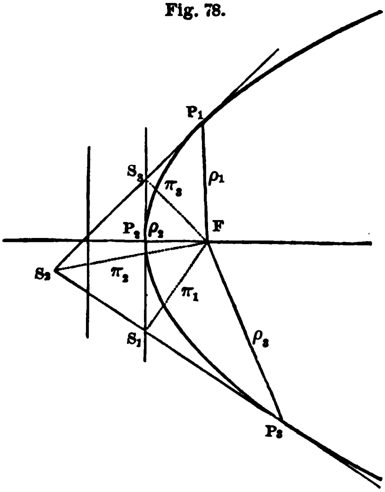
$$\pi_3 - \rho_1 = -\frac{p}{2}\alpha + \frac{y_1}{2}\beta - \left(\frac{y_1^2 - p^2}{2p}\alpha + y_1\beta\right)$$
$$= -\frac{y_1^2}{2p}\alpha - \frac{y_1}{2}\beta.$$

Operating with $S \,.\, \pi_3 \times$

$$S\pi_3(\pi_3 - \rho_1) = \frac{y_1^2}{4} - \frac{y_1^2}{4} = 0,$$

or, *a perpendicular from the focus on the tangent intersects it on the tangent at the vertex.*

Fig. 78.



Again, since π_3 is parallel to the normal at P_1, the latter may be written, from (i),

$$x\pi_3 = x\left(-\frac{p}{2}\alpha + \frac{y_1}{2}\beta\right) = z\alpha + y_1\beta;$$

whence

$$z = -x\frac{p}{2}, \quad y_1 = x\frac{y_1}{2},$$

or

$$x = 2, \qquad z = -p;$$

hence, *the subnormal is constant;* and *the normal is twice the perpendicular on the tangent from the focus.*

The normal at P_1 may be written

$$x\pi_3 = -z\alpha + \rho_1,$$

or

$$x\left(-\frac{p}{2}\alpha + \frac{y_1}{2}\beta\right) = -z\alpha + \frac{1}{2p}(y_1^2 - p^2)\alpha + y_1\beta;$$

whence, from (*b*),

$$x = 2, \quad \text{and} \quad z = \frac{1}{2p}(y_1^2 + p^2) = T\rho_1;$$

or, *the distance from the foot of the normal to the focus equals the radius vector to the point of contact, or the distance from the point of contact to the directrix, or the distance from the focus to the foot of the tangent.*

The portion of the tangent from its foot to the point of contact may be written $z\alpha + \rho_1$, in which z has just been found. Hence

$$z\alpha + \rho_1 = \frac{1}{2p}(y_1^2 + p^2)\alpha + \frac{1}{2p}(y_1^2 - p^2)\alpha + y_1\beta,$$

or

$$z\alpha + \rho_1 = \frac{y_1^2}{p}\alpha + y_1\beta, \tag{j}$$

the portion of the tangent from the foot of the focal perpendicular to the point of contact is

$$-\pi_3 + \rho_1 = \frac{p}{2}\alpha - \frac{y_1}{2}\beta + \frac{1}{2p}(y_1^2 - p^2)\alpha + y_1\beta,$$

or

$$-\pi_3 + \rho_1 = \frac{y_1^2}{2p}\alpha + \frac{y_1}{2}\beta, \tag{k}$$

or, comparing (*j*) and (*k*), *the tangent is bisected by the focal perpendicular*, and hence *the angles between the tangent and the axis and the tangent and the radius vector are equal*, and *the tangent bisects the angle between the diameter and radius vector to the point of contact.*

(k) is also the perpendicular from the focus on the normal, and shows that *the locus of the foot of the perpendicular from the focus on the normal is a parabola, whose vertex is at the focus of the given parabola and whose parameter is one-fourth that of the given parabola.*

88. The Ellipse.

1. Substituting in the general equation $\rho = x\alpha + y\beta$ the value of y from the equation of the ellipse referred to center and axes

$$a^2y^2 + b^2x^2 = a^2b^2,$$

we have

$$\rho = x\alpha + m^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}\beta \quad . \quad . \quad . \quad . \quad (220),$$

in which $m = \frac{b^2}{a^2}$ and α and β are unit vectors along the axes. For unit vectors along conjugate diameters, the equation of the ellipse becomes

$$\rho = x\alpha + m'^{\frac{1}{2}}(a'^2 - x^2)^{\frac{1}{2}}\beta \quad . \quad . \quad . \quad . \quad (221).$$

Again, if ϕ be the eccentric angle, the equation of the ellipse may be written in terms of a single scalar variable,

$$\rho = \cos\phi \,.\, \alpha + \sin\phi \,.\, \beta \quad . \quad . \quad . \quad . \quad (222).$$

2. From Eq. (220) we have, for the vector along the tangent,

$$\alpha - m^{\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}}x\beta = \alpha - \frac{m}{\sqrt{m}}\frac{x}{\sqrt{a^2 - x^2}}\beta = \alpha - \frac{mx}{y}\beta$$
$$= X(y\alpha - mx\beta)\,;$$

hence, for the equation of the tangent line,

$$\pi = x\alpha + y\beta + X(y\alpha - mx\beta) \quad . \quad . \quad . \quad (223)\,;$$

or, more simply, from Eq. (222), the vector-tangent is

$$-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta,$$

and the equation of the tangent is

$$\pi = \cos\phi \,.\, \alpha + \sin\phi \,.\, \beta + x(-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta), \text{ (224).}$$

Since $-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta$ is along the tangent, $\cos\phi \,.\, \alpha + \sin\phi \,.\, \beta$ and $-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta$ are vectors along conjugate diameters.

89. Examples *on the Ellipse.*

1. *The area of the parallelogram formed by tangents drawn through the vertices of any pair of conjugate diameters is constant.*

We have directly

$$\mathbf{TV}\left[2(\cos\phi \,.\, \alpha + \sin\phi \,.\, \beta)\, 2(-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta)\right]$$
$$= 4\,\mathbf{TV}\alpha\beta = a \text{ constant};$$

namely, the rectangle on the axes.

2. *The sum of the squares of conjugate diameters is constant, and equal to the sum of the squares on the axes.*

For, since $\mathbf{S}\alpha\beta = 0$,

$$(\cos\phi \,.\, \alpha + \sin\phi \,.\, \beta)^2 + (-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta)^2 = \alpha^2 + \beta^2.$$

3. *The eccentric angles of the vertices of conjugate diameters differ by* 90°.

The vector tangent at the extremity of

$$\rho = \cos\phi \,.\, \alpha + \sin\phi \,.\, \beta \qquad (a)$$

is

$$-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta.$$

This is also a vector along the diameter conjugate to ρ, and is seen to be the value of ρ when in (a) we write $\phi + 90°$ for ϕ.

4. *The eccentric angle of the extremity of equal conjugate diameters is* 45°, *and the diameters fall upon the diagonals of the rectangle on the axes.*

5. *The line joining the points of contact of tangents is parallel to the line joining the extremities of parallel diameters.*

6. *Tangents at right angles to each other intersect in the circumference of a circle.*

7. *If an ordinate* PD *to the major axis be produced to meet the circumscribed circle in* Q, *then*

$$\mathrm{QD} : \mathrm{PD} :: a : b.$$

8. *If an ordinate* PD *to the minor axis meets the inscribed circle in* Q, *then*

$$\mathrm{QD} : \mathrm{PD} :: b : a.$$

9. *Any semi-diameter is a mean proportional between the distances from the center to the points where it meets the ordinate of any point and the tangent at that point.*

For the point P (Fig. 82) we have

$$\rho = \cos\phi \cdot \alpha + \sin\phi \cdot \beta.$$

Also

$$\begin{aligned}\mathrm{OT} = x\mathrm{OP} &= \mathrm{OQ} + \mathrm{QT}\\ &= x(\cos\phi \cdot \alpha + \sin\phi \cdot \beta)\\ &= \cos\phi' \cdot \alpha + \sin\phi' \cdot \beta + t(-\sin\phi' \cdot \alpha + \cos\phi' \cdot \beta).\end{aligned}$$

Eliminating t,

$$x = \frac{1}{\cos(\phi - \phi')},$$

or

$$\mathrm{OT} = x\mathrm{OP} = \frac{1}{\cos(\phi - \phi')}\mathrm{OP}.$$

But

$$\begin{aligned}\mathrm{ON} = x'\mathrm{OP} &= \mathrm{OQ} + \mathrm{QN}\\ &= x'(\cos\phi \cdot \alpha + \sin\phi \cdot \beta)\\ &= \cos\phi' \cdot \alpha + \sin\phi' \cdot \beta + t'(-\sin\phi \cdot \alpha + \cos\phi \cdot \beta).\end{aligned}$$

Eliminating t',

$$x' = \cos(\phi - \phi'),$$

or

$$\mathrm{ON} = \cos(\phi - \phi')\mathrm{OP};$$

$$\therefore \mathrm{ON} \cdot \mathrm{OT} = \mathrm{OP}^2.$$

10. *To find the length of the curve.*

With the notation of Ex. 6, Art. 86, we obtain, from Eq. (222),

$$\rho' = -\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta,$$

$$\mathbf{T}\rho' = \sqrt{(a^2 - b^2)\sin^2\phi + b^2},$$

$$s - s_0 = \int_{\phi_0}^{\phi} \sqrt{(a^2 - b^2)\sin^2\phi + b^2},$$

which involves elliptic functions. If $a = b$, we have, for the circle, $s - s_0 = \int_{\phi_0}^{\phi} r = r(\phi - \phi_0)$.

From Eq. (220), we obtain

$$\rho' = \alpha - m^{\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}} x\beta,$$

$$\mathbf{T}\rho' = \sqrt{1 + \frac{m}{a^2 - x^2}x^2} = \frac{a}{\sqrt{a^2 - x^2}}\sqrt{1 - \frac{e^2x^2}{a^2}},$$

$$s - s_0 = \int_{x_0}^{x} \frac{a}{\sqrt{a^2 - x^2}}\sqrt{1 - \frac{e^2x^2}{a^2}},$$

which may be expanded and integrated; giving for the entire curve

$$2\pi a\left(1 - \frac{e^2}{2\,.\,2} - \frac{3e^4}{2\,.\,2\,.\,4\,.\,4} - \frac{3\,.\,3\,.\,5e^6}{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6} - \text{etc.}\right),$$

a converging series. If $e = 0$, we have, for the circle, $2\pi r$.

11. *To find the area of the ellipse.*

With the notation of Ex. 7, Art. 86,

$$\begin{aligned}\mathbf{TV}\rho\rho' &= \mathbf{TV}(\cos\phi \,.\, \alpha + \sin\phi \,.\, \beta)(-\sin\phi \,.\, \alpha + \cos\phi \,.\, \beta)\\ &= \mathbf{TV}(\cos^2\phi \,.\, \alpha\beta - \sin^2\phi \,.\, \beta\alpha) = \mathbf{TV}\alpha\beta;\end{aligned}$$

or, since $\widehat{\alpha\beta} = 90°$,

$$\tfrac{1}{2}\int_{\phi_0}^{\phi\frac{\pi}{2}} \mathbf{TV}\rho\rho' = \tfrac{1}{4}\pi ab.$$

The whole area is therefore πab.

90. The Hyperbola.

1. Let α and β be unit vectors parallel to the asymptotes. Then, from the equation,

$$xy = \frac{a^2 + b^2}{4} = m,$$

we have, for the equation of the hyperbola,

$$\rho = x\alpha + \frac{m}{x}\beta \quad . \quad . \quad . \quad . \quad . \quad (225);$$

or, if α and β are given vectors parallel to the asymptotes,

$$\rho = t\alpha + \frac{\beta}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226);$$

or, again, in terms of the eccentric angle,

$$\rho = \sec\phi \cdot \alpha + \tan\phi \cdot \beta \quad . \quad . \quad . \quad . \quad (227).$$

2. The equation of the tangent, obtained as usual, is from Eq. (226),

$$\rho = t\alpha + \frac{\beta}{t} + x\left(t\alpha - \frac{\beta}{t}\right) \quad . \quad . \quad . \quad . \quad (228),$$

where $t\alpha - \frac{\beta}{t}$ is a vector along the tangent.

91. Examples *on the Hyperbola.*

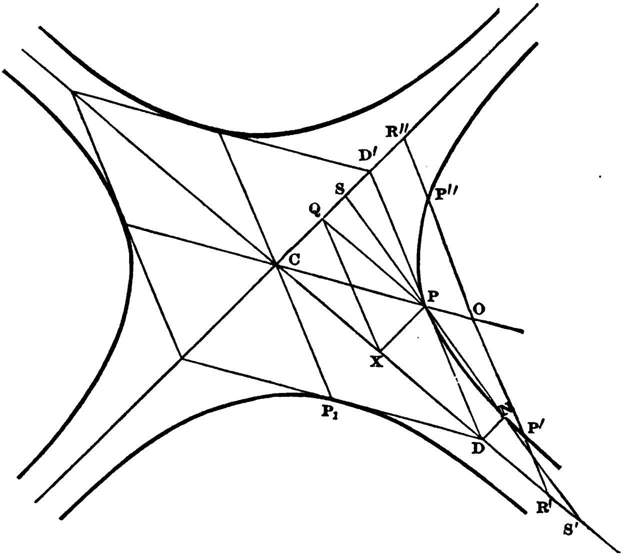
1. *If, when the hyperbola is referred to its asymptotes, one diagonal of a parallelogram whose sides are the coördinates is the radius vector, the other diagonal is parallel to the tangent.*

If (Fig. 79) $\text{CX} = t\alpha$, $\text{XP} = \frac{\beta}{t}$, then

$$\text{CP} = t\alpha + \frac{\beta}{t}, \quad \text{QX} = t\alpha - \frac{\beta}{t};$$

but $t\alpha - \frac{\beta}{t}$ is parallel to the tangent at P (Art. 90). $t\alpha + \frac{\beta}{t}$ and $t\alpha - \frac{\beta}{t}$ are evidently conjugate semi-diameters.

Fig. 79.



2. *A diameter bisects all chords parallel to the tangent at its vertex.*

Let (Fig. 79) CP be the diameter, t corresponding to the point P. The tangent at P is parallel to $t\alpha - \frac{\beta}{t}$ and $\text{CP} = t\alpha + \frac{\beta}{t}$. P'P'' being the parallel chord,

$$\text{CP}' = \text{CO} + \text{OP}' = x\left(t\alpha + \frac{\beta}{t}\right) + y\left(t\alpha - \frac{\beta}{t}\right).$$

Also, if t' correspond to P',

$$\text{CP}' = t'\alpha + \frac{\beta}{t'};$$

$$\therefore\ (x+y)t = t', \quad \frac{x-y}{t} = \frac{1}{t'},$$

or

$$x^2 - y^2 = 1.$$

Hence, for every point, as O, determined by x, there are two points P′ and P″, determined by the two corresponding values of y, which are equal with opposite signs.

3. *The tangent at* P_1 *to the conjugate hyperbola is parallel to* CP (Fig. 79).

4. *The portion of the tangent limited by the asymptotes is bisected at the point of contact.*

5. *If, from the point* D (Fig. 79), *where the tangent at* P *meets the asymptote,* DN *be drawn parallel to the other asymptote, then the portion of* PN *produced, which is limited by the asymptotes, is trisected at* P *and* N.

We have

$$\text{CN} = 2t\alpha + x\beta = t'\alpha + \frac{\beta}{t'} = 2t\alpha + \frac{\beta}{2t},$$

$$\text{CP} = t\alpha + \frac{\beta}{t};$$

$$\therefore\ \text{PN} = \text{CN} - \text{CP} = t\alpha - \frac{\beta}{2t},$$

and the equation of SS′ is

$$\rho = t\alpha + \frac{\beta}{t} + x\left(t\alpha - \frac{\beta}{2t}\right),$$

whence, for the points S, S′,

$$x = -1, \quad x = 2.$$

6. *The intercepts of the secant between the hyperbola and its asymptotes are equal.*

The vector along the tangent parallel to the secant is $t\alpha - \frac{\beta}{t}$. Hence (Fig. 79)

$$\text{CR}' = z\alpha = x\left(t\alpha + \frac{\beta}{t}\right) + y\left(t\alpha - \frac{\beta}{t}\right),$$

$$\text{CR}'' = z'\beta = x\left(t\alpha + \frac{\beta}{t}\right) + y'\left(t\alpha - \frac{\beta}{t}\right),$$

$$\therefore\ y = -y';$$

but OP″ = OP′ (Ex. 2), and therefore P″R″ = P′R′.

7. *If through any point* P'' (Fig. 79) *a line* $R''P'R'$ *be drawn in any direction, meeting the asymptotes in* R'' *and* R', *then*

$$P'R'' \cdot P'R' = PD'^2.$$

8. *If through* P', P'' (Fig. 79) *lines be drawn parallel to the asymptotes, forming a parallelogram of which* $P'P''$ *is one diagonal, the other diagonal will pass through the center.*

The vector from C to the farther extremity of the required diagonal is

$$t''\alpha + \frac{\beta}{t'} + (t' - t'')\alpha + \left(\frac{1}{t''} - \frac{1}{t'}\right)\beta = t'\alpha + \frac{\beta}{t''} = \frac{t'}{t''}\left(t''\alpha + \frac{\beta}{t'}\right).$$

But $t''\alpha + \frac{\beta}{t'}$ is the vector from C to the other extremity of the required diagonal.

9. *If the tangent at any point* P *meet the transverse axis in* T, *and* PN *be the ordinate of the point* P; *then*

$$CT \cdot CN = a^2,$$

C being the center and a the semi-transverse axis.

From Eq. (227), substituting in $CT = CP + PT$,

$$x \sec\phi \cdot \alpha = \sec\phi \cdot \alpha + \tan\phi \cdot \beta + y(\tan\phi \sec\phi \cdot \alpha + \sec^2\phi \cdot \beta);$$

$$\therefore\; x = \frac{1}{\sec^2\phi},$$

and

$$CT \cdot CN = (x \sec\phi \cdot \alpha)(\sec\phi \cdot \alpha) = \alpha^2,$$

or

$$CT \cdot CN = a^2.$$

10. *If the tangent at any point* P *meet the conjugate axis at* T', *and* PN' *be the ordinate to the conjugate axis, then*

$$CT' \cdot CN' = b^2,$$

C being the center and b the semi-conjugate axis.

92. The preceding examples on the conic sections involve directly the Cartesian forms. A method will now be briefly indicated peculiar to Quaternion analysis and independent of these forms.

1. The general form of an equation of the first degree, or as it may be called from analogy, a *linear* equation in quaternions, is

$$aqb + a'qb' + a''qb'' + \cdots\cdots = c,$$

or

$$\Sigma aqb = c, \tag{a}$$

in which q is an unknown quaternion, entering once, as a factor only, in each term, and a, b, a', b', ..., c are given quaternions. It may evidently be written

$$\Sigma\mathbf{S}aqb + \Sigma\mathbf{V}aqb = \mathbf{S}c + \mathbf{V}c,$$

whence

$$\Sigma\mathbf{S}aqb = \mathbf{S}c, \tag{b}$$

$$\Sigma\mathbf{V}aqb = \mathbf{V}b. \tag{c}$$

But

$$\mathbf{S}aqb = \mathbf{S}qba = \mathbf{S}q\mathbf{S}ba + \mathbf{S}\,.\,\mathbf{V}q\mathbf{V}ba,$$

and

$$\begin{aligned} \mathbf{V}aqb &= \mathbf{V}(\mathbf{S}a + \mathbf{V}a)(\mathbf{S}q + \mathbf{V}q)(\mathbf{S}b + \mathbf{V}b) \\ &= \mathbf{V}\,.\,\mathbf{S}q(\mathbf{S}a + \mathbf{V}a)(\mathbf{S}b + \mathbf{V}b) \\ &\qquad + \mathbf{V}(\mathbf{S}a\mathbf{V}q\mathbf{S}b + \mathbf{S}a\mathbf{V}q\mathbf{V}b + \mathbf{V}a\mathbf{V}q\mathbf{S}b + \mathbf{V}a\mathbf{V}q\mathbf{V}b) \\ &= \mathbf{S}q\mathbf{V}ab + \mathbf{V}(\mathbf{S}a\mathbf{S}b - \mathbf{S}a\mathbf{V}b + \mathbf{S}b\mathbf{V}a)\mathbf{V}q \\ &\qquad + \mathbf{V}\,.\,\mathbf{V}a\mathbf{V}q\mathbf{V}b + \mathbf{V}\,.\,\mathbf{V}a\mathbf{V}b\mathbf{V}q - \mathbf{V}\,.\,\mathbf{V}a\mathbf{V}b\mathbf{V}q \\ [\text{Eq. } (116)]\quad &= \mathbf{S}q\mathbf{V}ab + \mathbf{V}(\mathbf{S}a\mathbf{S}b - \mathbf{S}a\mathbf{V}b + \mathbf{S}b\mathbf{V}a - \mathbf{V}a\mathbf{V}b)\mathbf{V}q \\ &\qquad + 2\,\mathbf{V}a\mathbf{S}\,.\,\mathbf{V}q\mathbf{V}b \\ &= \mathbf{S}q\mathbf{V}ab + \mathbf{V}\,.\,a(\mathbf{K}b)\mathbf{V}q + 2\,\mathbf{V}a\mathbf{S}\,.\,\mathbf{V}q\mathbf{V}b. \end{aligned}$$

We have therefore, from (b) and (c),

$$\begin{aligned} \mathbf{S}c &= \mathbf{S}q\Sigma\mathbf{S}ba + \mathbf{S}\,.\,\mathbf{V}q\Sigma\mathbf{V}ba, \\ \mathbf{V}c &= \mathbf{S}q\Sigma\mathbf{V}ab + \Sigma\mathbf{V}\,.\,a(\mathbf{K}b)\mathbf{V}q + 2\,\Sigma\mathbf{V}a\mathbf{S}\,.\,\mathbf{V}q\mathbf{V}b, \end{aligned}$$

or, writing

$$\Sigma\mathbf{S}ab = d, \quad \Sigma\mathbf{V}ab = \delta, \quad \Sigma\mathbf{V}ba = \delta', \quad \mathbf{S}q = w, \quad \mathbf{V}q = \rho,$$

we obtain

$$\mathbf{S}c = wd + \mathbf{S}\rho\delta',$$
$$\mathbf{V}c = w\delta + \Sigma\mathbf{V} \,.\, a(\mathbf{K}b)\rho + 2\,\Sigma\mathbf{V}a\mathbf{S} \,.\, \rho\mathbf{V}b.$$

We may now eliminate w between these equations, obtaining

$$\mathbf{V}c \,.\, d - \mathbf{S}c \,.\, \delta = d\Sigma\mathbf{V}a(\mathbf{K}b)\rho - \delta\mathbf{S}\rho\delta' + d\,2\,\Sigma\mathbf{V}a\mathbf{S} \,.\, \rho\mathbf{V}b$$

which involves only the vector of the unknown quaternion q, and which, since $\mathbf{V}$ and Σ are commutative, may be written under the general form

$$\gamma = \mathbf{V}r\rho + \Sigma\beta\mathbf{S}\alpha\rho,$$

in which γ, α, α', ..., β, β', ... are known vectors, r a known quaternion, but ρ an unknown vector. This equation is the general form of a linear vector equation. The second member, being a linear function of ρ, may be written

$$\mathbf{V}r\rho + \Sigma\beta\mathbf{S}\alpha\rho = \phi\rho = \gamma \quad . \quad . \quad . \quad . \quad (229),$$

where $\phi\rho$ designates any linear function of ρ. If we define the inverse function ϕ^{-1} by the equation

$$\phi^{-1}(\phi\rho) = \rho,$$
$$\therefore\ \rho = \phi^{-1}\gamma,$$

the determination of ρ is made to depend upon that of ϕ^{-1}.

2. Without entering upon the solution of linear equations, it is evident on inspection that the function ϕ is distributive as regards addition, so that

$$\phi(\rho + \rho' + \cdots\cdots) = \phi\rho + \phi\rho' + \quad . \quad . \quad . \quad (230).$$

Also that, a being any scalar,

$$\phi a\rho = a\phi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231),$$

and

$$d\phi\rho = \phi d\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (232).$$

3. Furthermore, if we operate upon the form

$$\phi\rho = \Sigma\beta\mathbf{S}\alpha\rho + \mathbf{V}r\rho$$

with $S\,.\,\sigma\times$, σ being any vector whatever,

$$S\sigma\phi\rho = \Sigma S(\sigma\beta S\alpha\rho) + S\sigma(Vr\rho).$$

But

$$S(\sigma\beta S\alpha\rho) = S\sigma\beta S\alpha\rho = S\rho\alpha S\beta\sigma = S(\rho\alpha S\beta\sigma),$$

and

$$S(\sigma Vr\rho) = S[\sigma V(Sr + Vr)\rho] = SrS\sigma\rho + S\sigma(Vr)\rho$$
$$= SrS\rho\sigma - S\rho(Vr)\sigma = S[\rho V(Kr)\sigma].$$

Hence, if we designate by $\phi'\sigma$,

$$\phi'\sigma = \Sigma\alpha S\beta\sigma + V(Kr)\sigma,$$

a new linear function differing from ϕ by the interchange of the letters α and β, and Kr for r, we shall have, whatever the vectors ρ and σ,

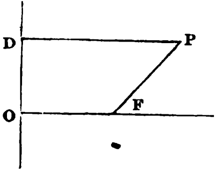
$$S(\sigma\phi\rho) = S(\rho\phi'\sigma).$$

Functions, which, like ϕ and ϕ', enjoy this property, are called conjugate functions. The function ϕ is said to be self-conjugate, that is, equal to its conjugate ϕ, when for any vectors ρ, σ,

$$S\sigma\phi\rho = S\rho\phi\sigma.$$

93. In accordance with Boscovich's definition, a conic section is the locus of a point so moving that the ratio of its distances from a fixed point and a fixed right line is constant.

Fig. 80.



1. Let F (Fig. 80) be the fixed point or focus, DO the fixed line or directrix, and P any point such that $\frac{\text{FP}}{\text{DP}} = e$, the constant ratio or eccentricity. Draw FO perpendicular to the directrix, and let $\text{FO} = \alpha$, $\text{OD} = y\gamma$, $\text{PD} = x\alpha$ and $\text{FP} = \rho$. By definition,

$$\frac{T\rho}{T(\text{PD})} = e,$$

or

$$\rho^2 = e^2x^2\alpha^2. \qquad (a)$$

Also

$$\rho + x\alpha = \alpha + y\gamma.$$

Operating with $S \, . \, \alpha \times$, we have, since $S\alpha\gamma = 0$,

$$S\alpha\rho + x\alpha^2 = \alpha^2,$$

or

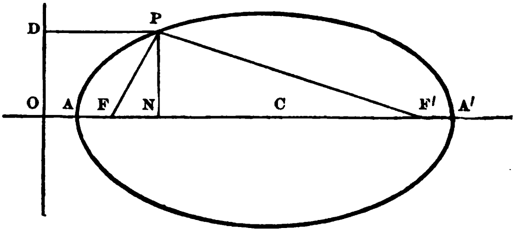
$$x^2\alpha^4 = (\alpha^2 - S\alpha\rho)^2.$$

Substituting in (a)

$$\alpha^2\rho^2 = e^2(\alpha^2 - S\alpha\rho)^2 \quad . \quad . \quad . \quad . \quad (233),$$

in which e may be less, greater than, or equal to unity, corresponding to the ellipse, hyperbola and parabola.

Fig. 81.



2. For the ellipse, Fig. 81, putting $\rho = x\alpha$ for the points A and A', we have

$$x = \frac{e}{1+e} \quad \text{and} \quad x = -\frac{e}{1-e},$$

or, since $\rho = x\alpha = x\text{FO}$,

$$\text{FA} = \frac{e}{1+e}\text{FO},$$

$$\text{FA}' = \frac{e}{1-e}\text{FO},$$

whence

$$\text{AA}' = 2a = \frac{2e}{1-e^2}\text{FO},$$

and therefore

$$\text{FO} = \frac{1-e^2}{e}a,$$

which furnish the well-known properties of the ellipse,

$$\begin{aligned} \text{FA} &= a(1-e), \\ \text{FA}' &= a(1+e), \\ \text{CF} &= ae, \\ \text{AO} &= \frac{1-e}{e}a, \\ \text{CO} &= \frac{a}{e}. \end{aligned}$$

3. Changing the origin to the center of the curve, let $\text{CF} = \alpha'$; then $\text{CP} = \rho'$ and $\rho = \rho' - \alpha'$, $\alpha' = \left(\frac{e}{1-e^2} - \frac{e}{1+e}\right)\alpha$; whence $\alpha = \frac{1-e^2}{e^2}\alpha'$. Substituting these values of ρ and α in

$$\alpha^2\rho^2 = e^2(\alpha^2 - \mathbf{S}\alpha\rho)^2,$$

remembering that $\alpha'^2 = -a^2e^2$, we obtain

$$a^2\rho'^2 + (\mathbf{S}\alpha'\rho')^2 = -a^4(1-e^2),$$

or, dropping the accents, C being the initial point,

$$a^2\rho^2 + (\mathbf{S}\alpha\rho)^2 = -a^4(1-e^2) \quad . \quad . \quad . \quad (234),$$

the equation of the ellipse in terms of the major axis with the origin at the center. If ρ coincides with the axes, $\mathbf{T}\rho = a$ or b, as it should.

4. Equation (234) may be deduced directly from Newton's definition, thus: let $\text{CF} = \alpha$ (Fig. 81) as before, F and F$'$ being the foci, and $\text{CP} = \rho$. Then

$$\text{FP} = \rho - \alpha, \quad \text{F}'\text{P} = \rho + \alpha,$$

and, by definition,

$$\text{FP} + \text{F}'\text{P} = 2a$$

as lines; or

$$\mathbf{T}(\rho - \alpha) + \mathbf{T}(\rho + \alpha) = 2a,$$

a being the semi-major axis. Whence

$$\sqrt{-(\rho-\alpha)^2} = 2a - \sqrt{-(\rho+\alpha)^2}.$$

Squaring

$$-\rho^2 + 2S\rho\alpha - \alpha^2 = 4a^2 - 4a\sqrt{-(\rho+\alpha)^2} - \rho^2 - 2S\rho\alpha - \alpha^2,$$

$$S\rho\alpha - a^2 = -a\sqrt{-(\rho+\alpha)^2}.$$

Squaring again

$$(S\rho\alpha)^2 - 2a^2 S\rho\alpha + a^4 = -a^2(\rho^2 + 2S\rho\alpha + \alpha^2),$$

$$a^2\rho^2 + (S\rho\alpha)^2 = -a^4 - a^2\alpha^2,$$

or, as before,

$$a^2\rho^2 + (S\rho\alpha)^2 = -a^4(1-e^2).$$

94. 1. The equation of the ellipse

$$a^2\rho^2 + (S\alpha\rho)^2 = -a^4(1-e^2)$$

may be put under the form

$$S\,.\,\rho\left[-\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)}\right] = 1;$$

or, in the notation of Art. 92, writing

$$-\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)} = \phi\rho,$$

the equation of the ellipse becomes

$$S\rho\phi\rho = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (235).$$

2. By inspection of the value of $\phi\rho$ it is seen that, when ρ coincides with either axis, ρ and $\phi\rho$ coincide.

Operating on $\phi\rho$ with $S\,.\,\sigma\times$, we have

$$S\sigma\phi\rho = -\frac{a^2 S\sigma\rho + S\sigma\alpha S\alpha\rho}{a^4(1-e^2)};$$

operating on $\phi\sigma = -\dfrac{a^2\sigma + \alpha S\alpha\sigma}{a^4(1-e^2)}$ with $S \,.\, \rho \times$, we have

$$S\rho\phi\sigma = -\frac{a^2 S\rho\sigma + S\rho\alpha S\alpha\sigma}{a^4(1-e^2)};$$

hence

$$S\rho\phi\sigma = S\sigma\phi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236),$$

and ϕ is self-conjugate.

3. Differentiating Equation (235), we have

$$S d\rho\phi\rho + S\rho d\phi\rho = 0,$$

$$S d\rho\phi\rho + S\rho\phi d\rho = 0, \qquad \text{[Eq. (232)]}$$

$$S\rho\phi d\rho + S\rho\phi d\rho = 2\, S\rho\phi d\rho = 0. \qquad \text{[Eq. (236)]}$$

If π be a vector to any point of the tangent line,

$$\pi = \rho + x d\rho,$$

whence

$$S\rho\phi(\pi - \rho) = S(\pi - \rho)\phi\rho = 0, \qquad (a)$$

or

$$S\pi\phi\rho = S\rho\phi\rho = S\rho\phi\pi = 1 \quad . \quad . \quad . \quad . \quad . \quad (237)$$

is the equation of the tangent line.

From (a) we see that $\phi\rho$ is a vector parallel to the normal at the point of contact, being parallel to ρ only when ρ coincides with the axes, as already remarked.

4. To transform the preceding equations into the usual Cartesian forms, let i be a unit vector along CA (Fig. 81), and j a unit vector perpendicular to it. If the coördinates of P are x and y, then, since $\alpha = $ CF,

$$\rho = xi + yj,$$

and

$$\phi\rho = -\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)} = -\frac{a^2(xi + yj) + aeiS \,.\, aei(xi + yj)}{a^4(1-e^2)}$$

$$= -\frac{a^2 xi\,(1 - e^2) + a^2 yj}{a^4(1 - e^2)},$$

or, since $1 - e^2 = \frac{b^2}{a^2}$

$$= -\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right);$$

$$\therefore \ S\rho\phi\rho = 1 = -S \,.\, (xi + yj)\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right),$$

and

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Again, if x' and y' are the coördinates of a point in the tangent,

$$\pi = x'i + y'j;$$

$$\therefore \ S\pi\phi\rho = 1 = -S(x'i + y'j)\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right),$$

and

$$a^2yy' + b^2xx' = a^2b^2.$$

The above applies to the hyperbola when $e > 1$, that is, when $1 - e^2 = -\frac{b^2}{a^2}$, giving the corresponding equations

$$a^2y^2 - b^2x^2 = -a^2b^2,$$
$$a^2yy' - b^2xx' = -a^2b^2.$$

95. Examples.

1. *To find the locus of the middle points of parallel chords.*

Let β be a vector along one of the chords, as RQ (Fig. 82), the length of the chord being $2y$, and let γ be the vector to its middle point; then

$$\rho = \gamma + y\beta \quad \text{and} \quad \rho = \gamma - y\beta$$

are vectors to points of the ellipse, and

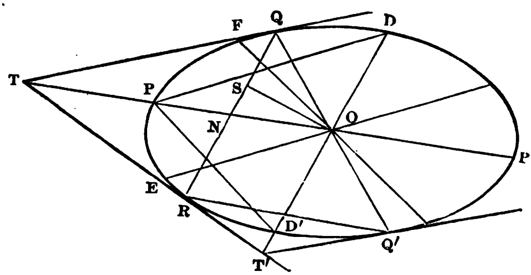
$$S(\gamma + y\beta)\phi(\gamma + y\beta) = 1,$$
$$S(\gamma - y\beta)\phi(\gamma - y\beta) = 1;$$

whence, expanding, subtracting, and applying Equation (236),

$$S\gamma\phi\beta = 0,$$

the equation of a straight line through the origin. Since $\phi\beta$ is parallel to the normal at the extremity of a diameter parallel to β, the locus is the diameter parallel to the tangent at that point.

Fig. 82.



2. *Equation of condition for conjugate diameters.*

Denote the diameter OP (Fig. 82) of the preceding problem, bisecting all chords parallel to β, by α. Then

$$S\alpha\phi\beta = 0,$$

or

$$S\beta\phi\alpha = 0.$$

In the latter, β is perpendicular to the normal $\phi\alpha$ at the extremity of α, and is therefore parallel to the tangent at that point; hence this is the equation of the diameter bisecting all chords parallel to α. Therefore, diameters which satisfy the equation $S\alpha\phi\beta = 0$ are conjugate diameters.

3. *Supplementary chords.*

Let PP′ (Fig. 82) and DD′ be conjugate diameters, and the chords PD, PD′ be drawn. Then, with the above notation,

$$DP = \alpha - \beta,$$
$$D'P = \alpha + \beta,$$

and

$$S(\alpha + \beta)\phi(\alpha - \beta) = S(\alpha + \beta)(\phi\alpha - \phi\beta)$$
$$= S(\alpha\phi\alpha - \alpha\phi\beta + \beta\phi\alpha - \beta\phi\beta).$$

But

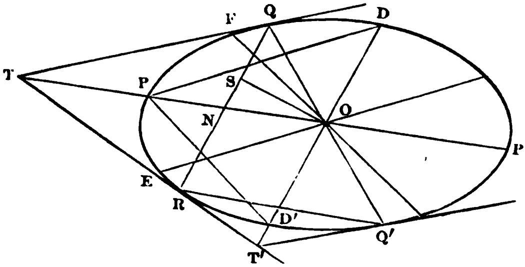
$$S\alpha\phi\alpha = 1, \quad S\beta\phi\beta = 1, \quad S\alpha\phi\beta = S\beta\phi\alpha;$$

$$\therefore \quad S(\alpha + \beta)\phi(\alpha - \beta) = 0.$$

Hence, if DP is parallel to a diameter, PD′ is parallel to its conjugate.

4. *If two tangents be drawn to the ellipse, the diameter parallel to the chord of contact and the diameter through the intersection of the tangents are conjugate.*

Fig 82.



Let TQ (Fig. 82) and TR be the tangents at the extremities of the chord parallel to β, and OT $= \pi$. Then

$$\text{OQ} = x\alpha + y\beta, \quad \text{OR} = x\alpha + y'\beta.$$

From the equation of the tangent $S\pi\phi\rho = 1$, we have

$$S\pi\phi(x\alpha + y\beta) = 1,$$
$$S\pi\phi(x\alpha + y'\beta) = 1.$$

Expanding and subtracting

$$S\pi\phi\beta = 0.$$

Hence, Ex. 2, π and β, or OP and OD, are conjugate. The locus of T for parallel chords is the diameter conjugate to the chord through the center.

5. *If* QOQ′ (Fig. 82) *be a diameter and* QR *a chord of contact, then is* Q′R *parallel to* OT.

RQ being parallel to β, and $\text{OQ}' = -\text{OQ}$, we have

$$\text{RQ} = 2y\beta, \quad \text{RQ}' = y\beta - x\alpha - x\alpha - y\beta\,;$$

whence, directly $\text{RQ}' = -2x\alpha$; as also $S\,\text{RQ}\,\phi\,\text{RQ}' = 0$, RQ and RQ′ being supplementary chords.

6. *The points in which any two parallel tangents as* Q′T′, QT (Fig. 82) *are intersected by a third tangent, as* TT′, *lie on conjugate diameters.*

The equation of RT′ is $S\pi\phi\rho = 1$, and that of Q′T′ is $S\pi'\phi\rho' = 1$. For the point T′, $\pi = \pi'$; whence, by subtraction,

$$S\pi\phi(\rho - \rho') = 0.$$

7. *Chord of contact.*

The equation of the tangent,

$$S\rho\phi\pi = 1,$$

is linear, and satisfied for both Q and R. Hence, writing σ for ρ as the variable vector, π being constant,

$$S\sigma\phi\pi = 1$$

is the equation of the chord of contact.

8. *To find the locus of* T *for all chords through a fixed point* (Fig. 82).

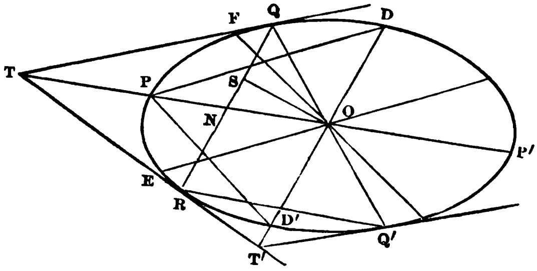
Let S be a fixed point of the chord RQ, so that $\text{OS} = \sigma =$ *a constant.* Then

$$S\sigma\phi\pi = S\pi\phi\sigma = 1,$$

a right line perpendicular to $\phi\sigma$, or parallel to the tangent at the extremity of OS, and the locus of T for all chords through S.

9. *Any semi-diameter is a mean proportional between the distances from the center to the points where it meets the ordinate of any point and the tangent at that point.*

Fig. 82.



OD (Fig. 82) and OP being still represented by β and α, let $\text{OT} = x'\alpha$ and $\text{OQ} = \rho = x\alpha + y\beta$. Then from the equation of the tangent, $S\pi\phi\rho = 1$, we obtain

$$Sx'\alpha\phi(x\alpha + y\beta) = 1;$$

whence, since $S\alpha\phi\beta = 0$,

$$xx'S\alpha\phi\alpha = 1,$$

or

$$xx' = 1;$$

$$\therefore\ xa \,.\, x'a = a^2,$$

or

$$\text{ON} \,.\, \text{OT} = \text{OP}^2.$$

10. *If* DD′ (Fig. 82) *and* PP′ *are conjugate diameters, then are* PD *and* PD′ *proportional to the diameters parallel to them.*

With the same notation

$$\text{DP} = \alpha - \beta, \qquad \text{D}'\text{P} = \alpha + \beta,$$

whence

$$\text{OE} = m(\alpha - \beta), \quad \text{OF} = n(\alpha + \beta).$$

From the equation of the ellipse

$$Sm(\alpha-\beta)\phi m(\alpha-\beta)=1, \qquad (a)$$

and

$$Sn(\alpha+\beta)\,\phi n\,(\alpha+\beta)=1. \qquad (b)$$

Now, from (a), since $S\beta\phi\beta=S\alpha\phi\alpha=1$ and $S\beta\phi\alpha=S\alpha\phi\beta=0$,

$$2m^2=1.$$

Similarly, from (b),

$$2n^2=1;$$

$$\therefore\ m=n.$$

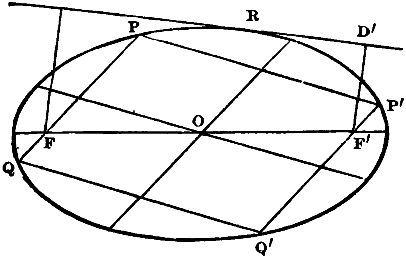
Also

$$\begin{aligned}\text{DP}:\text{D}'\text{P}&::T(\alpha-\beta):T(\alpha+\beta)\\&::Tm(\alpha-\beta):Tn(\alpha+\beta)\\&::\text{OE}:\text{OF}.\end{aligned}$$

11. *The diameters along the diagonals of the parallelogram on the axes are conjugate; and the same is true of diameters along the diagonals of any parallelogram whose sides are the tangents at the extremities of conjugate diameters.*

12. *Diameters parallel to the sides of an inscribed parallelogram are conjugate.*

Fig. 83.



Let the sides of the parallelogram (Fig. 83) be

$$\text{PP}'=\alpha,\quad \text{PQ}=\beta,$$

and let

$$\text{OP}=\rho,\quad \text{OQ}=\rho'.$$

Then

$$\text{OP}'=\rho+\alpha,\quad \text{OQ}'=\rho'+\alpha,\quad \rho'-\rho=\beta.$$

From the equation of the ellipse, $S\rho\phi\rho=1$, we have for Q′ and P′

$$S(\rho'+\alpha)\phi(\rho'+\alpha)=1,$$
$$S(\rho+\alpha)\phi(\rho+\alpha)=1\,;$$

whence, since $S\rho\phi\rho=S\rho'\phi\rho'=1$,

$$2\,S\alpha\phi\rho'+S\alpha\phi\alpha=0,$$
$$2\,S\alpha\phi\rho+S\alpha\phi\alpha=0.$$

Subtracting

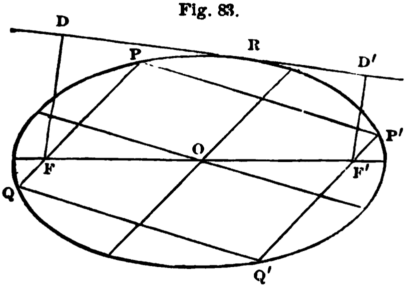
$$S\rho'\phi\alpha-S\rho\phi\alpha=0,$$

or

$$S(\rho'-\rho)\phi\alpha=S\beta\phi\alpha=0.$$

13. *The rectangle of the perpendiculars from the foci on the tangent is constant, and equal to the square of the semi-conjugate axis.*

Fig. 83.



Let the tangent be drawn at R (Fig. 83) and $OR=\rho$. Then $\phi\rho$ is parallel to the normal at R, that is, to the perpendiculars FD, F′D′. Hence, OF being α,

$$OD'=x'\phi\rho-\alpha,$$
$$OD=\alpha+x\phi\rho,$$

which, since D and D′ are on the tangent, in $S\pi\phi\rho=1$ give

$$S(x'\phi\rho-\alpha)\phi\rho=1,$$
$$S(\alpha+x\phi\rho)\phi\rho=1,$$

or

$$x'(\phi\rho)^2=1+S\alpha\phi\rho,$$
$$x\,(\phi\rho)^2=1-S\alpha\phi\rho\,;$$

whence

$$T x'\phi\rho = \mathrm{F'D'} = T\frac{1+S\alpha\phi\rho}{\phi\rho},$$

$$T x\phi\rho = \mathrm{FD} = T\frac{1-S\alpha\phi\rho}{\phi\rho},$$

and

$$\mathrm{FD}\times\mathrm{F'D'} = T\frac{1-(S\alpha\phi\rho)^2}{(\phi\rho)^2}.$$

But

$$(\phi\rho)^2 = \left(-\frac{a^2\rho+\alpha S\alpha\rho}{a^4(1-e^2)}\right)^2 = \frac{a^2(a^2\rho^2)+2a^2(S\alpha\rho)^2+\alpha^2(S\alpha\rho)^2}{a^8(1-e^2)^2},$$

or, substituting $a^2\rho^2$ from Equation (234) and $\alpha^2=-a^2e^2$,

$$=\frac{(S\alpha\rho)^2-a^4}{a^6(1-e^2)}.$$

Also

$$1-(S\alpha\phi\rho)^2 = 1-\left[\frac{a^2S\alpha\rho+\alpha^2S\alpha\rho}{a^4(1-e^2)}\right]^2 = \frac{a^4-(S\alpha\rho)^2}{a^4}.$$

$$\therefore\ \mathrm{FD}\times\mathrm{F'D'} = T\frac{a^4-(S\alpha\rho)^2}{a^4}\,\frac{a^6(1-e^2)}{(S\alpha\rho)^2-a^4} = a^2(1-e^2) = b^2.$$

14. *The foot of the perpendicular from the focus on the tangent is in the circumference of the circle described on the major axis.*

To prove this we have to show that the line OD (Fig. 83) is equal to a. Now

$$\begin{aligned}\mathrm{OD} &= \alpha + x\phi\rho\\ &= \alpha + \frac{\phi\rho(1-S\alpha\phi\rho)}{(\phi\rho)^2}\end{aligned}$$

from the preceding example. Hence

$$\begin{aligned}(\mathrm{OD})^2 &= \alpha^2 + \frac{2S\alpha\phi\rho(1-S\alpha\phi\rho)}{(\phi\rho)^2} + \frac{(1-S\alpha\phi\rho)^2}{(\phi\rho)^2}\\ &= \alpha^2 + \frac{1-(S\alpha\phi\rho)^2}{(\phi\rho)^2} = -a^2e^2 + \frac{a^4-(S\alpha\rho)^2}{a^4}\,\frac{a^6(1-e^2)}{(S\alpha\rho)^2-a^4}\\ &= -a^2e^2 - a^2(1-e^2) = -a^2;\end{aligned}$$

$$\therefore\ \mathrm{OD} = a.$$

The Parabola.

96. 1. Resuming Equation (233) and making $e=1$, the equation of the parabola is

$$\alpha^2\rho^2 = (\alpha^2 - S\alpha\rho)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (238),$$

which may be written

$$\frac{\rho^2 + 2\,S\alpha\rho - \alpha^{-2}(S\alpha\rho)^2}{a^2} = 1,$$

or

$$S\rho\left[\frac{\rho + 2\,\alpha - \alpha^{-2}\alpha S\alpha\rho}{a^2}\right] = 1\,;$$

in which, if we put

$$\phi\rho = \frac{\rho - \alpha^{-1}S\alpha\rho}{a^2},$$

we have for the equation of the parabola

$$S\rho(\phi\rho + 2\,\alpha^{-1}) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (239),$$

and, as in the case of the ellipse,

$$S\sigma\phi\rho = S\rho\phi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (240).$$

Operating on $\phi\rho$ by $S \,.\, \alpha \times$, we obtain

$$S\alpha\phi\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241);$$

hence, $\phi\rho$ *is a perpendicular to the axis.*

Operating on $\phi\rho$ by $S \,.\, \rho \times$

$$S\rho\phi\rho = \frac{\rho^2 - \alpha^{-2}(S\alpha\rho)^2}{a^2} = a^2(\phi\rho)^2 \quad . \quad . \quad . \quad (242).$$

2. Differentiating Equation (239), we have

$$2\,S\rho\phi d\rho + 2\,Sd\rho\alpha^{-1} = 0.$$

For any point of the tangent line to which the vector is π,

$$\pi = \rho + xd\rho,$$

from which, substituting $d\rho$ in the above,

$$S\rho\phi(\pi-\rho)\ \ +S(\pi-\rho)\alpha^{-1}=0,$$
$$S(\rho\phi\pi-\rho\phi\rho+\pi\alpha^{-1}-\rho\alpha^{-1})=0; \qquad (a)$$

or, since $S\rho\phi\rho=1-2\,S\rho\alpha^{-1}$ [Eq. (239)],

$$S\pi\phi\rho-1+2\,S\rho\alpha^{-1}+S\pi\alpha^{-1}-S\rho\alpha^{-1}=0,$$

whence

$$S\pi(\phi\rho+\alpha^{-1})+S\rho\alpha^{-1}=1 \quad . \quad . \quad . \quad . \quad (243),$$

the equation of the tangent line.

3. From (a) we obtain

$$S(\pi-\rho)(\phi\rho+\alpha^{-1})=0;$$

or, since $\pi-\rho$ is a vector along the tangent,

$$\phi\rho+\alpha^{-1}$$

is in the direction of the normal.

4. If σ be a vector to any point of the normal, the equation of the normal will be

$$\sigma=\rho+x(\phi\rho+\alpha^{-1}) \quad . \quad . \quad . \quad . \quad . \quad (244).$$

5. The Cartesian form of Equation (239) is obtained by making

$$\rho=xi+yj, \quad \alpha=\text{FO (Fig. 80)}=-pi;$$

$$\therefore\ \phi\rho=\frac{xi+yj-\dfrac{xpi}{p}}{-p^2}=-\frac{yj}{p^2};$$

whence, Equation (239) becomes

$$\frac{y^2}{p^2}-\frac{2x}{p}=1;$$

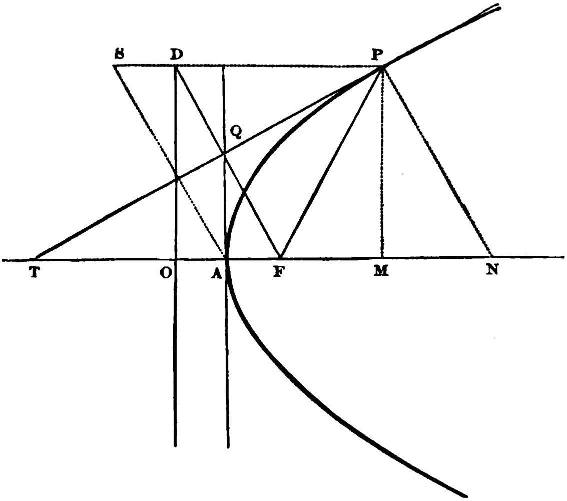
$$\therefore\ y^2=2px+p^2,$$

the equation of the parabola referred to the focus.

97. Examples.

1. *The subtangent is bisected at the vertex.*

Fig. 84.



We have (Fig. 84) $\text{FT} = x\alpha$, which in the equation of the tangent

$$S\pi(\phi\rho + \alpha^{-1}) + S\rho\alpha^{-1} = 1$$

gives

$$Sx\alpha(\phi\rho + \alpha^{-1}) + S\alpha^{-1}\rho = 1.$$

But $S\alpha\phi\rho = 0$; hence

$$x + S\alpha^{-1}\rho = 1; \qquad (a)$$

multiplying by α

$$x\alpha + \alpha S\alpha^{-1}\rho = \alpha,$$

$$(x - \tfrac{1}{2})\alpha = \alpha - \tfrac{1}{2}\alpha - \alpha S\alpha^{-1}\rho$$

$$= \tfrac{1}{2}\alpha - \alpha S\alpha^{-1}\rho,$$

$$\text{AT} = -\text{AF} - \alpha S\alpha^{-1}\rho.$$

But the value of $\phi\rho$ gives

$$\alpha^2\phi\rho = \rho - \alpha^{-1}S\alpha\rho;$$

and, since $\phi\rho$ is a vector along MP and $\alpha^{-1}S\alpha\rho$ a vector along FM, from $\rho = \text{FM} + \text{MP}$ we have

$$\text{FM} = \alpha^{-1}S\alpha\rho = \alpha S\alpha^{-1}\rho, \qquad (b)$$

$$\text{MP} = \alpha^2\phi\rho\,; \qquad (c)$$

$$\therefore\ \text{AT} = -\text{AF} - \text{FM} = -\text{AM},$$

or, as lines,

$$\text{AT} = \text{AM}.$$

2. *The distances from the focus to the point of contact and the intersection of the tangent with the axis are equal.*

$$x\alpha = \alpha - \alpha S\alpha^{-1}\rho,$$

or (Fig. 84),

$$\begin{aligned}(\text{FT})^2 &= (\alpha - \alpha S\alpha^{-1}\rho)^2\\ &= (\alpha - \alpha^{-1}S\alpha\rho)^2\\ &= \frac{(\alpha^2 - S\alpha\rho)^2}{\alpha^2}\\ &= \rho^2; \qquad \text{[Eq. (238)]}\end{aligned}$$

$$\therefore\ \text{FP} = \text{FT}.$$

3. *The subnormal is constant and equal to half the parameter.*

The vector-normal being $\phi\rho + \alpha^{-1}$ (Art. 96, 3), we have (Fig. 84)

$$\text{PN} = z(\phi\rho + \alpha^{-1})\,;$$

but

$$\begin{aligned}\text{PN} &= \text{PM} + \text{MN}\\ &= -\alpha^2\phi\rho + x\alpha\,; \qquad \text{[Ex. 1, (c)]}\end{aligned}$$

$$\therefore\ z(\phi\rho + \alpha^{-1}) = -\alpha^2\phi\rho + x\alpha,$$

$$z = -\alpha^2 = x\alpha^2,$$

or

$$x = -1, \quad x\alpha = -\alpha\,;$$

or, the distances MN and FO are equal, and the subnormal $= p$, a constant.

4. *The perpendicular from the focus on the tangent intersects it on the tangent at the vertex, and* $\text{AQ} = \frac{1}{2}\text{MP}$ (Fig. 84).

Since (Ex. 2) FP = FT = PD, FD is perpendicular to PT or parallel to PN. Otherwise:

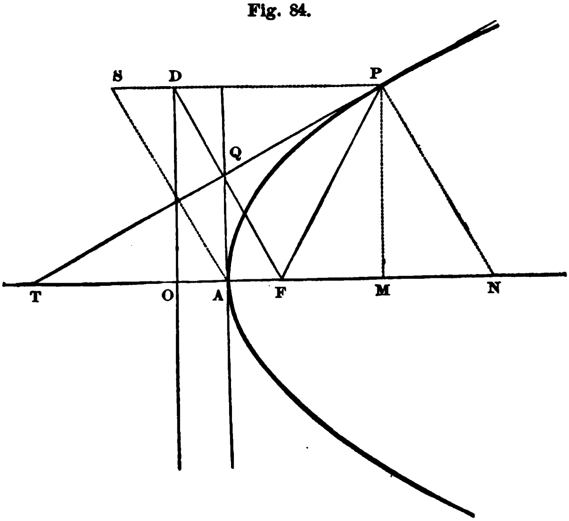
$$\text{NP} = -z(\phi\alpha + \alpha^{-1}) = \alpha^2(\phi\rho + \alpha^{-1}) \qquad \text{(Ex. 3)}$$
$$= \alpha^2\phi\rho + \alpha = \text{MP} + \text{FO}, \qquad [\text{Ex. 1, } (c)]$$

But

$$\alpha^2\phi\rho + \alpha = \text{FO} + \text{OD} = \text{FD}.$$
$$\tfrac{1}{2}\text{FD} = \text{FQ} = \tfrac{1}{2}\alpha^2\phi\rho + \tfrac{1}{2}\alpha$$
$$= \tfrac{1}{2}\alpha^2\phi\rho + \text{FA};$$
$$\therefore\ \tfrac{1}{2}\alpha^2\phi\rho = \text{AQ} = \tfrac{1}{2}\text{MP}.$$

5. *To find the locus of the intersection of the perpendicular from the vertex on the tangent and the diameter produced through the point of contact.*

Fig. 84.



Let FS = σ (Fig. 84) be a vector to a point of the locus. Then

$$\text{FS} = \text{FA} + \text{AS} = \text{FP} + \text{PS},$$
$$\sigma = \tfrac{1}{2}\alpha + z(\phi\rho + \alpha^{-1}) = \rho + x\alpha.$$

Operating with $\times$ S . $\phi\rho$, then, since $S\alpha\phi\rho = 0$ [Eq. (241)],

$$z(\phi\rho)^2 = S\rho\phi\rho = a^2(\phi\rho)^2; \qquad \text{[Eq. (242)]}$$

$$\therefore\ z = a^2,$$

and

$$\sigma = \tfrac{1}{2}\alpha + a^2(\phi\rho + \alpha^{-1}) = \tfrac{3}{2}\alpha + a^2\phi\rho,$$

or

$$\sigma - \tfrac{3}{2}\alpha = a^2\phi\rho.$$

Operating with $\times$ S . α

$$S(\sigma - \tfrac{3}{2}\alpha)\alpha = 0,$$
$$S\sigma\alpha = -\tfrac{3}{2}(T\alpha)^2,$$

or [Eq. (180)], the locus is a right line perpendicular to the axis and $\frac{3}{2}p$ distant from the focus.

6. *To find the locus of the intersection of the tangent and the perpendicular from the vertex.*

If the origin be taken at the vertex, then since $\phi\rho + \alpha^{-1}$ is a vector along the normal, the equation of the locus will be

$$\pi = x(\phi\rho + \alpha^{-1}). \qquad (a)$$

To eliminate x, operate with S . α $\times$ which gives

$$x = S\alpha\pi, \quad \text{whence} \quad S\alpha^{-1}\pi = -\frac{x}{a^2}.$$

To eliminate ρ, the equation of the tangent, $S\pi(\phi\rho + \alpha^{-1}) + S\rho\alpha^{-1} = 1$, for the new origin becomes

$$S\left(\pi + \frac{\alpha}{2}\right)(\phi\rho + \alpha^{-1}) + S\rho\alpha^{-1} = 1,$$

or

$$2\,S\pi\phi\rho + 2\,S\alpha^{-1}\pi + 2\,S\alpha^{-1}\rho = 1.$$

Operating on (a) with $\times$ S . $\phi\rho$, whence $S\pi\phi\rho = x(\phi\rho)^2$, the preceding equation becomes

$$2x(\phi\rho)^2 - \frac{2x}{a^2} + 2\,S\alpha^{-1}\rho = 1. \qquad (b)$$

Also [Eq. (242)] $S\rho\phi\rho = \alpha^2(\phi\rho)^2$, which, in the equation of the parabola $S\rho(\phi\rho + 2\alpha^{-1}) = 1$, gives

$$\alpha^2(\phi\rho)^2 + 2S\alpha^{-1}\rho = 1. \qquad (c)$$

Whence, from (*b*) and (*c*), by subtraction,

$$(\phi\rho)^2 = \frac{2x}{2a^2x + a^4}.$$

But, from (*a*),

$$(\phi\rho)^2 = \frac{\pi^2 - 2xS\pi\alpha^{-1} + x^2\alpha^{-2}}{x^2} = \frac{\pi^2}{x^2} + \frac{1}{a^2}.$$

Equating these values of $(\phi\rho)^2$, and substituting the value of x,

$$2\pi^2 S\alpha\pi - \alpha^2\pi^2 + (S\alpha\pi)^2 = 0,$$

which is the equation of the locus required. To transform to Cartesian coördinates, make

$$\pi = xi + yj, \quad \text{and} \quad \alpha = ai,$$

whence

$$\pi^2 = -(x^2 + y^2), \quad S\alpha\pi = -ax, \quad \alpha^2 = -a^2,$$

and

$$y^2 = \frac{x^3}{\frac{a}{2} - x},$$

the equation of the cissoid to the circle whose diameter is the distance from the vertex to the directrix.

7. *If* PP′ (Fig. 75) *be a focal chord, and* PA, PA′ *produced meet the directrix in* D′, D, *then will* PD *and* P′D′ *be parallel to* AF.

$$AD' = -x AP = AO + OD',$$

or

$$x\left(\frac{\alpha}{2} - \rho\right) = \frac{\alpha}{2} + y\gamma.$$

Operating with $S \,.\, \alpha \times$

$$x(\alpha^2 - 2S\alpha\rho) = \alpha^2. \qquad (a)$$

Now FP $= \rho$ and FP$' = -x'\rho$ are vectors to points on the curve, and hence satisfy its equation. Whence [Eq. (238)]

$$\alpha^2\rho^2 = (\alpha^2 - \mathrm{S}\alpha\rho)^2,$$
$$x'^2\alpha^2\rho^2 = (\alpha^2 + x'\mathrm{S}\alpha\rho)^2;$$
$$\therefore\ x'^2(\alpha^2 - \mathrm{S}\alpha\rho)^2 = (\alpha^2 + x'\mathrm{S}\alpha\rho)^2;$$

or

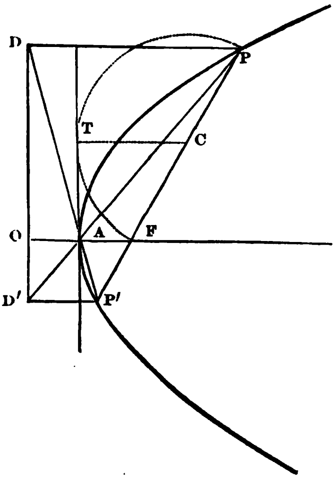
$$x'(\alpha^2 - \mathrm{S}\alpha\rho) = \alpha^2 + x'\mathrm{S}\alpha\rho,$$
$$\therefore\ x'(\alpha^2 - 2\,\mathrm{S}\alpha\rho) = \alpha^2.$$

Hence, comparing with (a),

$$x = x';$$

or, the sides produced of the triangle APF are cut proportionately, and therefore D$'$P$'$ is parallel to AF.

Fig. 75 (*bis*).



8. *If, with a diameter equal to three times the focal distance, a circle be described with its center at the vertex, the common chord bisects the line joining the focus and vertex.*

The equation of the curve being

$$\alpha^2\rho^2 = (\alpha^2 - \mathrm{S}\alpha\rho)^2, \qquad (a)$$

that of the circle whose center is A (Fig. 75), referred to F, is of the form [Eq. (210)]

$$\mathrm{T}(\rho - \gamma) = \mathrm{T}\beta,$$

or, by condition,

$$\mathrm{T}\left(\rho - \frac{\alpha}{2}\right) = \mathrm{T}\tfrac{3}{4}\alpha;$$
$$\therefore\ \left(\rho - \frac{\alpha}{2}\right)^2 = \tfrac{9}{16}\alpha^2,$$
$$\rho^2 = \mathrm{S}\alpha\rho + \tfrac{5}{16}\alpha^2,$$

which, in (a), gives

$$\mathrm{S}\alpha\rho = \frac{\alpha^2}{4},$$

which is the proposition.

98. The Cycloid.

1. Let α and β be vectors along the base and axis of the cycloid and $T\beta = T\alpha = r$, the radius of the generating circle. Then, for any point P of the curve,

$$\begin{aligned} x &= r\theta - r\sin\theta = r(\theta - \sin\theta), \\ y &= r \quad - r\cos\theta = r(1 - \cos\theta), \end{aligned}$$

and the equation of the cycloid is

$$\rho = (\theta - \sin\theta)\alpha + (1 - \cos\theta)\beta.$$

2. The vector along the tangent is

$$(1 - \cos\theta)\alpha + \sin\theta \cdot \beta,$$

and the equation of the tangent is

$$\pi = (\theta - \sin\theta)\alpha + (1 - \cos\theta)\beta + t[(1 - \cos\theta)\alpha + \sin\theta \cdot \beta].$$

3. The vector from P to the lower extremity of the vertical diameter of the generating circle through P is

$$\text{PC} = -(1 - \cos\theta)\beta + \sin\theta \cdot \alpha,$$

and, from the above expression, for the vector-tangent PT,

$$S(\text{PC} \cdot \text{PT}) = 0;$$

hence PC is perpendicular to the tangent, or the normal passes through the foot of the vertical diameter of the generating circle for the point to which the normal is drawn, and the tangent passes through the other extremity.

4. If, through P, a line be drawn parallel to the base, intersecting the central generating circle in Q, show that $\text{PQ} = r(\pi - \theta) = \text{arc QA}$, A being the upper extremity of the axis.

5. With the notation of Ex. 6, Art. 86,

$$\begin{aligned}\rho' &= (1-\cos\theta)\alpha + \sin\theta \cdot \beta,\\ \rho'^2 &= -[(1-\cos\theta)^2 + \sin^2\theta]r^2,\\ \mathrm{T}\rho' &= r\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta} = r\sqrt{2-2\cos\theta}\\ &= 2r\sin\tfrac{1}{2}\theta;\\ s - s_0 &= \int_{2\pi}^{0} 2r\sin\tfrac{1}{2}\theta = [4r\cos\tfrac{1}{2}\theta]_{2\pi}^{0} = 8r,\end{aligned}$$

the length of the entire curve.

6. With the notation of Ex. 7, Art. 86,

$$\begin{aligned}\mathrm{TV}\rho\rho' &= \mathrm{TV}[(\theta-\sin\theta)\sin\theta \cdot \alpha\beta + (1-\cos\theta)^2\beta\alpha]\\ &= \mathrm{TV}[(\theta\sin\theta - \sin^2\theta - (1-\cos\theta)^2]\alpha\beta\\ &= r^2(\theta\sin\theta + 2\cos\theta - 2).\\ A - A_0 &= r^2\int_{2\pi}^{0}(\theta\sin\theta + 2\cos\theta - 2)\\ &= \left[\frac{r^2}{2}(\sin\theta - \theta\cos\theta + 2\sin\theta - 2\theta)\right]_{2\pi}^{0}\\ &= \left[\frac{r^2}{2}(3\sin\theta - \theta\cos\theta - 2\theta)\right]_{2\pi}^{0} = 3\pi r^2,\end{aligned}$$

the whole area of the curve.

99. Elementary Applications to Mechanics.

1. If b be the magnitude of any force acting in a known direction, the force, as having magnitude and direction, may be represented by the vector symbol β, which is independent of the point of application of the force. In order, completely, to define the force with reference to any origin O, the vector $\mathrm{OA} = \alpha$, to its point of application A, must also be given. For concurring forces, whose magnitudes are b_i' b_i'' ·····, we have, for the resultant, $\beta = \Sigma\beta_i'$ which is true, whether the forces are complanar or not, and is the *theorem of the polygon of forces* extended. For two forces, $\beta = \beta' + \beta''$; whence $\beta^2 = \beta'^2 + \beta''^2 + 2\mathrm{S}\beta'\beta''$, or

$b^2 = b'^2 + b''^2 + 2b'b'' \cos\theta$, which is the *theorem of the parallelogram of forces.* For any number of concurring forces, the condition of equilibrium will be $\Sigma\beta' = 0$. For a particle constrained to move on a plane curve whose equation is $\rho = \phi(t)$, $d\rho$ being in the direction of the tangent, since the resultant of the extraneous forces must be normal to the curve for equilibrium, we have

$$\mathbf{S}d\rho\Sigma\beta' = \mathbf{S}d\rho\beta = 0. \tag{a}$$

2. If $\mathrm{OA}' = \alpha'$, and β' is a force acting at A', then $\mathbf{TV}\alpha'\beta' = a'b' \sin\theta$ is the numerical value of the *moment of the couple* β' at A' and $-\beta'$ at O. Representing, as usual, the couple by its axis, its vector symbol will be $\mathbf{V}\alpha'\beta'$. If $-\beta'$ act at some point other than the origin, as C', and $\mathrm{OC}' = \gamma'$, the couple will be denoted by $\mathbf{V}(\alpha' - \gamma')\beta'$. From this vector representation of couples, it follows that *their composition is a process of vector addition;* hence the *resultant couple* is $\Sigma\mathbf{V}(\alpha' - \gamma')\beta'$, and, for equilibrium, $\Sigma\mathbf{V}(\alpha' - \gamma')\beta' = 0$. If the couples are in the same or parallel planes, their axes are parallel and $\mathbf{T}\Sigma = \Sigma\mathbf{T}$. Since $\alpha' - \gamma'$ is independent of the origin, *the moment of the couple is the same for all points.* Since $\mathbf{V}(\alpha' - \gamma')\beta' = \mathbf{V}\alpha'\beta' - \mathbf{V}\gamma'\beta'$, *the moment of a couple is the algebraic sum of the moments of its component forces.* If the forces are concurring, and α' is the vector to their common point of application, $\Sigma\mathbf{V}\alpha'\beta' = \mathbf{V}\Sigma\alpha'\beta' = \mathbf{V}\alpha'\Sigma\beta' = \mathbf{V}\alpha'\beta$, or *the moment of the resultant about any point is the sum of the moments of the component forces.* When the origin is on the resultant, α' coincides with β' in direction, and $\mathbf{V}\alpha'\beta = 0$; or *the algebraic sum of the moments about any point of the resultant is zero.* If a single force β' acts at A', we may, as usual, introduce two equal and opposite forces at the origin, or at any other point C', and thus replace $\beta'_{\mathrm{A}'}$ by β'_0 and $\mathbf{V}\alpha'\beta'$, or by $\beta'_{\mathrm{C}'}$ and $\mathbf{V}(\alpha' - \gamma')\beta'$. If ζ be a unit vector along any axis OZ through the origin, then the moment of β' acting at A', with reference *to the axis* OZ, will be $-\mathbf{S}\beta'\alpha'\zeta$, or $-\mathbf{S}\,.\,\zeta\mathbf{V}\beta'\alpha'$. If β' and ζ are in the same plane, in which case they either intersect or are parallel; or, if the axis passes through A', there will be no moment: in these cases, α', β' and ζ are complanar, and $-\mathbf{S}\beta'\alpha'\zeta = 0$.

3. If the forces are parallel, their resultant $\beta = \Sigma\beta' = \Sigma b'\text{U}\beta' = \text{U}\beta\Sigma b'$; and, therefore, for equilibrium, $\Sigma\text{T}\beta' = \Sigma b' = 0$. The moment of a force with reference to any axis oz through the origin being $-\text{S}\beta'\alpha'\zeta$, and the moment of the resultant being equal to the sum of the moments of the components, we have $\text{S}\beta\alpha\zeta = \Sigma\text{S}\beta'\alpha'\zeta$, which, for parallel forces, becomes $\text{S}(\Sigma b' \, . \, \text{U}\beta \, . \, \alpha\zeta) = \text{S}(\text{U}\beta\Sigma b'\alpha' \, . \, \zeta)$, which, being true for any axis, is satisfied for $\Sigma b' \, . \, \alpha = \Sigma b'\alpha'$;

$$\therefore \ \alpha = \frac{\Sigma b'\alpha'}{\Sigma b'}, \tag{b}$$

which is independent of $\text{U}\beta$, and hence is the vector to the *center of parallel forces*. When $\Sigma b' = 0$, the above equations give $\beta = 0$ and $\alpha = \infty$, the system reducing to a couple. For a system of particles whose weights are w', w'' ·····, we have the vector to the *center of gravity* $\alpha = \frac{\Sigma w'\alpha'}{\Sigma w'}$. From this equation, $\Sigma w'(\alpha - \alpha') = 0$; whence, if the particles are equal, *the sum of the vectors from the center of gravity to each particle is zero;* and, if unequal, and the length of each vector is increased proportionately to the weight of each particle, their sum is zero. For equal particles, $\alpha = \frac{w'\Sigma\alpha'}{\Sigma w'}$, or *the center of gravity of a system of equal particles is the mean point* (Art. 18) *of the polyedron of which the particles are the vertices.* For a continuous body whose weight is w, volume v, and density D at the extremity of α, $\alpha = \frac{\Sigma \text{D} dv\alpha'}{\Sigma \text{D} dv}$, in which Σ may be replaced by the integral sign if the density is a known function of the volume. For a homogeneous body, $\alpha = \frac{\Sigma dv\alpha'}{\Sigma dv}$, which is applicable to lines, surfaces or solids, v representing a line, area or volume. Thus, for a plane curve $\rho = \phi(t) = \alpha'$, $dv = ds = \text{T}d\rho = \text{T}\phi'(t)dt$ and

$$\alpha = \frac{\int \phi(t)\text{T}\phi'(t)\,dt}{\int \text{T}\phi'(t)\,dt}. \tag{c}$$

4. *General conditions of equilibrium of a solid body.* Let the forces $\beta', \beta'' \cdots\cdots$, act at the points A$'$, A$'' \cdots\cdots$ of a solid body, and $\text{OA}' = \alpha'$, $\text{OA}'' = \alpha'' \cdots\cdots$. Replacing each force by an equal one at the origin and a couple, the given system will be equivalent to a system of concurring forces at the origin and a system of couples. Hence, for equilibrium,

$$\Sigma\beta' = 0, \qquad (d)$$
$$\Sigma V\alpha'\beta' = 0. \qquad (e)$$

Let ξ be the vector to *any* point X. Then, from (d), $V \,.\, \xi\Sigma\beta' = 0$, and therefore, from (e), $V \,.\, \xi\Sigma\beta' = \Sigma V\alpha'\beta'$; whence

$$\Sigma V\beta'\alpha' - \Sigma V\beta'\xi = \Sigma V\beta'(\alpha' - \xi) = 0. \qquad (f)$$

Conversely, ξ being a vector to *any* point, the resultant couple, for equilibrium, is $\Sigma V(\alpha' - \xi)\beta' = 0$; $\therefore$ $\Sigma V\alpha'\beta' = 0$ and $\Sigma\beta' = 0$. Therefore (f) is the necessary and sufficient condition of equilibrium.

This condition may be otherwise expressed by the principle of virtual moments. Let $\delta', \delta'' \cdots\cdots$ be the displacements. Then the virtual moment of β' is $-S\beta'\delta'$; and, for equilibrium, $\Sigma S\beta'\delta' = 0$. This equation involves (d) and (e). Thus, if the displacement corresponds to a simple translation, $\delta' = \delta'' = \delta'''$ $=$ etc. $=$ *a constant*, and we may write $\Sigma S\beta'\delta' = S\delta\Sigma\beta' = 0$; whence, since δ is real, $\Sigma\beta' = 0$. Again, if the displacement corresponds to a rotation about an axis ζ, ζ being a unit vector along the axis,

$$\alpha' = \zeta^{-1}\zeta\alpha' = \zeta^{-1}(S\zeta\alpha' + V\zeta\alpha') = -\zeta S\zeta\alpha' - \zeta V\zeta\alpha',$$

the last term being a vector perpendicular to the axis. For a rotation about this axis through an angle θ, this term becomes $-\zeta^{\frac{2\theta}{\pi}}\zeta V\zeta\alpha' = -\zeta\cos\theta\, V\zeta\alpha' + \sin\theta\, V\zeta\alpha'$, and α' becomes

$$\alpha'_1 = -\zeta S\zeta\alpha' - \zeta\cos\theta\, V\zeta\alpha' + \sin\theta\, V\zeta\alpha',$$

which, for an infinitely small displacement,

$$= -\zeta S\zeta\alpha' - \zeta V\zeta\alpha' + \theta V\zeta\alpha'.$$

Placing the scalar factor under the vector sign and writing ζ simply for $\theta\zeta$, to denote the indefinitely short vector along oz,

$$\alpha' + \delta' = \alpha' + \mathbf{V}\zeta\alpha';$$

or, $\delta' = \mathbf{V}\zeta\alpha'$. Hence $\Sigma\mathbf{S}\beta'\delta' = \Sigma\mathbf{S}\beta'\mathbf{V}\zeta\alpha' = \mathbf{S}\zeta\Sigma\mathbf{V}\alpha'\beta'$; or, since ζ is not zero, $\Sigma\mathbf{V}\alpha'\beta' = 0$.

5. *Illustrations.*

(1) Three concurrent forces, represented in magnitude and direction by the medials of any triangle, are in equilibrium. (See Ex. 2, Art. 18.)

(2) If three concurring forces are in equilibrium, they are complanar. By condition, $\beta' + \beta'' + \beta''' = 0$. Operating with $\mathbf{S}\,.\,\beta'\beta''\times$, we have $\mathbf{S}\beta'\beta''\beta''' = 0$.

(3) In the preceding case, operating with $\mathbf{V}\,.\,\beta'\times$, we have $\mathbf{V}\beta'\beta'' + \mathbf{V}\beta'\beta''' = 0$; whence, since the forces are complanar, $\mathbf{TV}\beta'\beta'' = \mathbf{TV}\beta'\beta'''$ or $b'b''\sin(\beta', \beta'') = b'b'''\sin(\beta', \beta''')$. A similar relation may be found for any two of the forces; whence

$$b' : b'' : b''' :: \sin(\beta'', \beta''') : \sin(\beta', \beta''') : \sin(\beta', \beta'').$$

(4) If two forces are represented in magnitude and position by two chords of a semicircle drawn from a point on the circumference, the diameter through the point represents the resultant.

(5) A weight, w', rests on the arc of a vertical plane curve, and is connected, by a cord passing over a pulley, with another weight, w''. Find the relation between the weights for equilibrium.

(a) Let the curve be a parabola, and the pulley at the focus. Then, from Eq. (a) of this article, the equation of the curve being $\rho = \frac{1}{2p}(y^2 - p^2)\alpha + y\beta$, we have

$$\mathbf{S}\left(\frac{y}{p}\alpha + \beta\right)\left(-w'\alpha + \frac{y\beta + x\alpha}{r}w''\right) = 0,$$

in which $r=$ radius vector. Hence

$$-\frac{y}{p}w'+\frac{y}{p}\frac{x}{r}w''+\frac{y}{r}w''=0,$$

or, since $r=x+p$, $w'=w''$. Hence, if the weights are equal, equilibrium will exist at all points of the curve.

(*b*) Let the curve be a circle and the pulley at a distance m from the curve on the vertical diameter produced. With the origin at the highest point of the circle, $\rho=x\alpha+\sqrt{2\mathrm{R}x-x^2}\beta$. Hence, r being the distance of the pulley from w',

$$\mathrm{S}\left(\frac{\mathrm{R}-x}{y}\beta+\alpha\right)\left(w'\alpha-\frac{y\beta+(m+x)\alpha}{r}w''\right)=0;$$

$$\therefore\ w''=\frac{rw'}{\mathrm{R}+m}.$$

(*c*) Let w be placed on the concave arc of a vertical circle, and acted upon by a repulsive force varying inversely as the square of the distance from the lowest point of the circle. To find the position of equilibrium. The origin being at the lowest point of the circle, and r the distance required, let p be the intensity of the force at a unit's distance; then $\frac{p}{r^2}$ will be its intensity for any distance r, and

$$\mathrm{S}\left(\alpha+\frac{\mathrm{R}-x}{y}\beta\right)\left(\frac{x\alpha+y\beta}{r}\frac{p}{r^2}-w\alpha\right)=0;$$

whence

$$r=\sqrt[3]{\frac{p\mathrm{R}}{w}}.$$

(*d*) Let w' rest on a right line inclined at an angle θ to the horizontal, and connected with w'' by a cord passing over a pulley at the upper end of the line. Find the relation between the weights. With the origin at the lower end of the line, its equation is $\rho=x\alpha$. If β is in the direction of w', then $\mathrm{S}\alpha(w'\beta+w''\alpha)=0$; $\therefore\ w''=w'\sin\theta$.

(6) To find the center of gravity of three equal particles at the vertices of a triangle. A, B, C being the vertices, the vector

from A to the center of gravity of the weights at A and B is $\frac{1}{2}\text{AB} = \text{AD}$. The vector to the center of gravity of the three weights is $\frac{1}{3}(\text{AB} + \text{AC}) = \frac{1}{2}\text{AB} + x\text{DC} = \frac{1}{2}\text{AB} + x(-\frac{1}{2}\text{AB} + \text{AC})$; $\therefore\ x = \frac{1}{3}$, and the required point is the center of gravity of the triangle.

(7) Find the center of gravity of the perimeter of a triangle.

(8) Find the center of gravity of four equal particles at the vertices of a tetraedron.

(9) Show that the center of gravity of four equal particles at the angular points of any quadrilateral is at the middle point of the line joining the middle points of a pair of opposite sides.

(10) The center of gravity of the triangle formed by joining the extremities of perpendiculars, erected outwards, at the middle points of any triangle, and proportional to the corresponding sides, coincides with that of the original triangle. Let ABC be the triangle, $\text{BC} = 2\alpha$, $\text{CA} = 2\beta$ and ϵ a vector perpendicular to the plane of the triangle. Then, if m is the given ratio, B the initial point, and R_1, R_2, R_3 the extremities of the perpendiculars to BC, CA, AB, respectively,

$$\text{BR}_1 = \alpha + m\epsilon\alpha,\quad \text{BR}_2 = 2\alpha + \beta + m\epsilon\beta,\quad \text{BR}_3 = \alpha + \beta - m\epsilon(\alpha + \beta);$$

$$\therefore\ \tfrac{1}{3}(\text{BR}_1 + \text{BR}_2 + \text{BR}_3) = \tfrac{1}{3}(4\alpha + 2\beta) = \tfrac{1}{3}[2\alpha + 2(\alpha + \beta)].$$

(11) To find the center of gravity of a circular arc. The equation of the circle $\rho = r(\cos\theta\,.\,\alpha + \sin\theta\,.\,\beta)$, gives $d\rho = r(-\sin\theta\,.\,\alpha + \cos\theta\,.\,\beta)d\theta$;

$$\therefore\ \alpha_1 = \frac{\int \phi(\theta)\mathrm{T}\phi'(\theta)d\theta}{\int \mathrm{T}\phi'(\theta)d\theta} = \frac{\int r(\cos\theta\,.\,\alpha + \sin\theta\,.\,\beta)d\theta}{\int d\theta}.$$

For an arc of 90°, integrating between the limits $\frac{\pi}{2}$ and 0, $\alpha_1 = \frac{2r}{\pi}(\alpha + \beta)$, the distance from the center being $\frac{2r}{\pi}\sqrt{2}$; which may be obtained directly also by integrating between the limits $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. For a semicircumference or arc of 60°, we have, in like manner, $\frac{2r}{\pi}$ and $\frac{3r}{\pi}$.

(12) If α, β, γ are the vector edges of any tetraedron, the origin being at the vertex, then $\rho-\alpha$, $\beta-\gamma$, $\alpha-\beta$ are lines of the base, ρ being any vector to its plane. Hence this plane is represented by $S(\rho-\alpha)(\beta-\gamma)(\alpha-\beta)=0$; $\therefore\ S\rho(V\alpha\beta+V\gamma\alpha+V\beta\gamma)-S\alpha\beta\gamma=0$. If δ be the vector perpendicular on the base,

$$\delta = x(V\alpha\beta+V\gamma\alpha+V\beta\gamma)=\frac{S\alpha\beta\gamma}{V\alpha\beta+V\beta\gamma+V\gamma\alpha},$$

and, taking the tensors,

$$T(V\alpha\beta+V\beta\gamma+V\gamma\alpha)=\frac{6\times\text{vol.}}{\text{alt.}}=2\,\text{area base}.$$

But $V\alpha\beta+V\beta\gamma+V\gamma\alpha+V\beta\alpha+V\gamma\beta+V\alpha\gamma=0$, in which the last terms are twice the *vector* areas of the plane faces. The sum of the vector areas of all the faces is therefore zero. Since any polyedron may be divided into tetraedra by plane sections, whose vector areas will have the same numerical coefficient, but have opposite signs two and two, *the sum of the vector areas of any polyedron is zero.* These vector areas represent the pressures on the faces of a polyedron immersed in a perfect fluid subjected to no external forces. For rotation, since the points of application of these pressures are the centers of gravity of the faces, to which the vectors are

$$\tfrac{1}{3}(\alpha+\beta+\gamma),\quad \tfrac{1}{3}(\beta+\alpha),\quad \tfrac{1}{3}(\gamma+\beta),\quad \tfrac{1}{3}(\alpha+\gamma),$$

we have the couples

$$\begin{aligned}&-\tfrac{1}{6}V\{(\alpha+\beta+\gamma)(V\alpha\beta+V\beta\gamma+V\gamma\alpha)+(\alpha+\beta)V\beta\alpha+(\beta+\gamma)\\&\qquad V\gamma\beta+(\gamma+\alpha)V\alpha\gamma\}\\&=-\tfrac{1}{6}V(\alpha V\beta\gamma+\beta V\gamma\alpha+\gamma V\alpha\beta),\end{aligned}$$

since $\alpha V\alpha\beta+\alpha V\beta\alpha=0$, etc. But, Equation (123), this sum is zero. Hence there is no rotation.

100. Miscellaneous Examples.

1. In Fig. 58, F, A and K are collinear.

2. In Fig. 58, $AD^2 - AE^2 = AB^2 - AC^2$.

3. In Fig. 13, if the lines from the vertices of the parallelogram through O and P are angle-bisectors, OMHP is a rectangle.

4. If the corresponding sides of two triangles are in the same ratio, the triangles are similar.

5. β, α, γ being the vector sides of a plane triangle, if $\beta = \alpha + \gamma$, show that $b^2 = c^2 - ca \cos B + ab \cos C$.

6. The sides BC, CA, AB of a triangle are produced to D, E, F, so that $CD = m BC$, $AE = n CA$, $BF = p AB$. Find the intersections Q_1, Q_2, Q_3 of EB, FC; FC, DA; DA, EB.

7. In any right-angled triangle, four times the sum of the squares of the medials to the sides about the right angle is equal to five times the square of the hypothenuse.

8. If ABC be any triangle, M its mean point, and O any point in space, then

$$AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2) - (3\,OM)^2.$$

9. If ABCD be any quadrilateral, M its mean point, and O any point in space, then

$$AB^2 + BC^2 + CD^2 + DA^2$$
$$= 4(OA^2 + OB^2 + OC^2 + OD^2) - (4\,OM)^2 - AC^2 - BD^2.$$

10. If ABC be any triangle, and C′, B′, A′ the middle points of AB, AC, CB, then, O being any point in space,

$$AB^2 + BC^2 + CA^2 = 4(OA^2 + OB^2 + OC^2) - 4(OB'^2 + OC'^2 + OA'^2).$$

11. If ABC be any triangle and M its mean point, then

$$AB^2 + BC^2 + CA^2 = 3(AM^2 + BM^2 + CM^2).$$

12. Points P, Q, R, S are taken in the sides AB, BC, CD, DA of a parallelogram, so that $AP = m AB$, $BQ = m BC$, etc. Show that PQRS is a parallelogram whose mean point coincides with that of ABCD.

13. The sides of any quadrilateral are divided equably at P, Q, R, S, and the points of division joined in succession. If PQRS is a parallelogram, the original quadrilateral is a parallelogram,

14. The middle points of the three diagonals of a complete quadrilateral are collinear.

15. If any quadrilateral be divided into two quadrilaterals by any cutting line, the centers of the three are collinear.

16. If a circle be described about the mean point of a parallelogram as a center, the sum of the squares of the lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

17. A quadrilateral possesses the following property: any point being taken, and four triangles formed by joining this point with the angular points of the figure, the centers of gravity of these triangles lie in the circumference of a circle. Prove that the diagonals of this quadrilateral are at right angles to each other.

18. The sum of the vector perpendiculars from A, B, C, ... on any line through their mean point is zero.

19. a, b, c are the three adjacent edges of a rectangular parallelopiped. Show that the area of the triangle formed by joining their extremities is $\frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

20. Given the co-ordinates of A, B, C, D referred to rectangular axes. Find the volume of the pyramid O—ABCD, O being the origin.

21. Any plane through the middle points of two opposite edges of a tetraedron bisects the latter.

22. The chord of contact of two tangents to a circle drawn from the same point is perpendicular to the line joining that point with the center.

23. If two circles cut each other and from one point of section a diameter be drawn to each circle, the line joining their extremities is parallel to the line joining their centers, and passes through the other point of section.

24. The square of the sum of the diameters of two circles, tangent at a common point, is equal to the sum of the squares of any two common chords through the point of tangency, at right angles to each other.

25. T is any point without a circle whose centre is C; from T draw two tangents TP, TQ, also any line cutting the circle in V, and PQ in R; draw CS perpendicular to TV. Then $SR . ST = SV^2$.

26. If a series of circles, tangent at a common point, are cut by a fixed circle, the lines of section meet in a point.

27. In Ex. 26, the intersections of the pairs of tangents to the fixed circle, at the points of section, lie in a straight line.

28. If three given circles are cut by any circle, the lines of section form a triangle, the loci of whose angular points are right lines perpendicular to the lines joining the centers of the given circles.

29. The three loci of Ex. 28 meet in a point.

30. Given the base of an isosceles triangle, to find the locus of the vertex.

31. Find the locus of the center of a circle which passes through two given points.

32. Find the locus of the center of a sphere of given radius, tangent to a given sphere.

33. The locus of the point from which two circles subtend equal angles is a circle, or a right line.

34. Given the base of a triangle, and *m* times the square of one side plus *n* times the square of the other, to find the locus of the vertex.

35. Given the base and the sum of the squares of the sides of a triangle, to find the locus of the vertex.

36. In Ex. 35, given the difference of the squares, to find the locus.

37. OB and OA are any two lines, and MP is a line parallel to OB. Find the locus of the intersection of OQ and BQ drawn parallel to AP and OP, respectively.

38. From a fixed point P, on the surface of a sphere, chords PP′, PP″, ... are drawn. Find the locus of a point O on these chords, such that PP′. PO $= m^2$.

39. A line of constant length moves with its extremities on two straight lines at right angles to each other. Find the locus of its middle point.

40. Find the locus of a point such that if straight lines be drawn to it from the four corners of a square, the sum of their squares is constant.

41. Find the locus of a point the square of whose distance from a given point is proportional to its distance from a given line.

42. Find the locus of the feet of perpendiculars from the origin on planes cutting off pyramids of equal volume from three rectangular co-ordinate axes.

43. Given the base of a triangle and the ratio of the sides, to find the locus of the vertex.

44. Show that $\mathrm{V}\alpha\rho\mathrm{V}\rho\beta = (\mathrm{V}\alpha\beta)^2$ is the equation of a hyperbola whose asymptotes are parallel to α and β.

45. Find the point on an ellipse the tangent to which cuts off equal distances on the axes.

46. A and B are two similar, similarly situated, and concentric ellipses; C is a third ellipse similar to A and B, its center being on the circumference of B, and its axes parallel to those of A and B: show that the chord of intersection of A and B is parallel to the tangent to B at the center of C.